MATH4023
Tutorial 7 - Cauchy Integral Formula
April 7/9, 2020 Notes by Ivan So

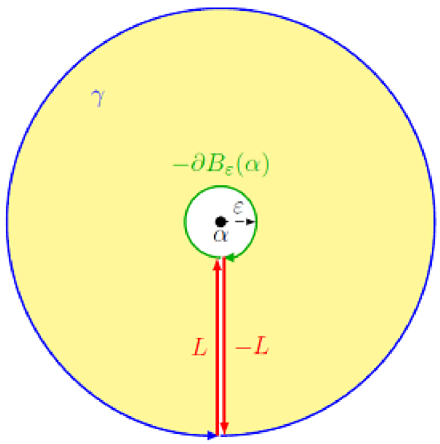
1 Review

Theorem 1.1. Suppose f is *holomorphic* on a open set Ω and contains the closure $\overline{D}$ of the disc D. Then for any $z \in D$,

$$f(z) = \frac{1}{2\pi i}\int_{\partial\overline{D}} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Proof sketch: Consider a *keyhole* domain excluding the point z can apply Cauchy-Goursat's theorem.

- Consider the keyhole domain:



- Applying Cauchy-Goursat's theorem, then

$$\oint_\gamma \frac{f(z)}{z-\alpha}dz = \oint_{|z-\alpha|=\epsilon} \frac{f(z)}{z-\alpha}dz = \boxed{\oint_{|z-\alpha|=\epsilon} \frac{f(z)-f(\alpha)}{z-\alpha}dz} + \oint_{|z-\alpha|=\epsilon} \frac{f(\alpha)}{z-\alpha}dz$$

- The first integral vanishes.
- Second integral can be obtained from direct evaluation, which is $2\pi i f(\alpha)$.

Remark 1.2. The *keyhole* argument of Cauchy integral formula enable us to integrate functions with multiple singularity without the method of *partial fraction*. The idea is closely related to method of **residue calculus** (see later lectures).

2 Problems

1. **True or False**

   (a) The keyhole domain is not simply-connected.
   False. Notice that although there is a "hole" in the keyhole domain, the "hole" is obtained through limiting. □

   (b) The reason behind the vanishing of the first term (boxed in the review section) when we apply the Cauchy-Goursat's theorem in the proof of the Cauchy's integral formula is complex differentiability.
   True. By differentiability, $\lim_{z\to a}\frac{f(z)-f(a)}{z-a}$ exists. So over $B_\epsilon(a)$, $\frac{f(z)-f(a)}{z-a}$ is bounded. Using integral approximation and taking $\epsilon \to 0$, we see that the boxed term vanishes. □

2. Let f be holomorphic on $\overline{B}(z_0, b)$. Show that

$$\frac{1}{\pi b^2}\int\int_{\overline{B}_b(z_0)} f(x+iy)dydx = f(z_0).$$

Solution: Let $g(z) = f(z+z_0)$. A linear change of variables shows that

$$\int\int_{\overline{B}_b(z_0)} f(x+iy)dxdy = \int\int_{\overline{B}_b(0)} g(x+iy)dxdy.$$

So we can just assume $z_0 = 0$. If $0 < r < b$ and C_r denotes the circle centered at the origin of radius r. Cauchy's integral formula implies

$$f(0) = \frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta)}{\zeta}d\zeta.$$

Parametrize C_r by $re^{i\theta}$ with $\theta \in [0, 2\pi]$, then

$$\begin{aligned} f(0) &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(re^{i\theta})}{re^{i\theta}}ire^{i\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})d\theta \\ \Rightarrow \int_0^b f(0)rdr &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^b f(re^{i\theta})rdrd\theta \\ \Rightarrow f(0) &= \frac{1}{\pi b^2}\int\int_{B_b(0)} f(x+iy)dydx \qquad \text{(cartesian to polar)} \end{aligned}$$

□

3. Let f be a holomorphic function on $B_{R_0}(0)$.

   (a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then
$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\zeta})\mathrm{Re}\left(\frac{Re^{i\zeta}+z}{Re^{i\zeta}-z}\right)d\zeta.$$

   (b) Show that
$$\mathrm{Re}\left(\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}\right) = \frac{R^2-r^2}{R^2-2Rr\cos\gamma+r^2}.$$

Solution:

(a) Using the Cauchy's integral formula,
$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{C_R}\frac{f(\zeta)}{\zeta-z}d\zeta - 0\\
&= \frac{1}{2\pi i}\int_{C_R}\left(\frac{f(\zeta)}{\zeta-z}-\frac{f(\zeta)}{\zeta-w}\right)d\zeta && (w\notin B_R(0))\\
&= \frac{1}{2\pi i}\int_0^{2\pi}\left(\frac{f(Re^{i\varphi})}{Re^{i\varphi}-z}-\frac{f(Re^{i\varphi})}{Re^{i\varphi}-w}\right)\cdot iRe^{i\varphi}d\varphi\\
&= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\left(\frac{Re^{i\varphi}}{Re^{i\varphi}-z}-\frac{\overline{z}}{\overline{z}-Re^{-i\varphi}}\right)d\varphi\\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi}{\xi-z}+\frac{\overline{z}}{\overline{\xi}-\overline{z}}\right)d\varphi && (\xi := Re^{i\varphi})\\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi\overline{\xi}-z\overline{z}}{(\xi-z)(\overline{\xi}-\overline{z})}\right)d\varphi\\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi+z}{\xi-z}+\frac{\overline{\xi}+\overline{z}}{\overline{\xi}-\overline{z}}\right)d\varphi = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)d\varphi
\end{aligned}$$

(b)
$$\begin{aligned}
\mathrm{Re}\left(\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}\right) &= \frac{1}{2}\left(\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}+\frac{Re^{-i\gamma}+r}{Re^{-i\gamma}-r}\right)\\
&= \frac{1}{2}\frac{R^2-rRe^{i\gamma}+rRe^{-i\gamma}-r^2+R^2-Rre^{-i\gamma}-r^2-Rre^{-i\gamma}}{R^2+r^2-2rR\cos\gamma}\\
&= \frac{R^2-r^2}{R^2+r^2-2rR\cos\gamma}
\end{aligned}$$

□

4. Show that there is no function f holomorphic in $\Omega = \mathbb{C}\setminus[-1,1]$ such that $f(z)^2 = z$.

Solution: Assume such f exists, then for any $z \oint \Omega$, taking complex derivative on both sides give
$$2f(z)f'(z) = 1.$$

Dividing both sides by $f(z)^2$,
$$\begin{aligned}
\frac{1}{2z} &= \frac{f'(z)}{f(z)}\\
\Rightarrow \oint_{C_1}\frac{1}{2z}dz &= \oint_{C_1}\frac{f'(z)}{f(z)}dz\\
\pi i &= \int_0^{2\pi}\frac{ie^{it}f'(e^{it})}{f(e^{it})}dt\\
&= \int_0^{2\pi}\frac{[f(e^{it})]'}{f(e^{it})}dt && (*)
\end{aligned}$$

Define $g(s) = f(e^{is}) \exp\left[-i\int_0^s \frac{f'(e^{it})}{f(e^{it})} e^{it} dt\right]$, then $g'(s) \equiv 0$, so $g(s)$ is a constant with $g(2\pi) = g(0)$. So

$$f(1)\left(1 - \exp\left[-i\int_0^{2\pi} \frac{f'(e^{it})e^{it}}{f(e^{it})} dt\right]\right) = 0.$$

Since $2f'(z)f(z) = 1 \neq 0$, $f(1) \neq 0$. Therefore

$$\int_0^{2\pi} \frac{f'(e^{it})e^{it}}{f(e^{it})} dt \in 2\pi\mathbb{Z}.$$

Now, since LHS of $(*)$ is in $i\pi\mathbb{Z}$ but RHS $2\pi i\mathbb{Z}$, a contradiction. Therefore such f cannot exists. $\square$

5. (a) Show that the association $f \mapsto f'/f$ (for a holomorphic f) sends product to sum.

   (b) If $P(z) = (z - a_1)\cdots(z - a_n)$, what is P'/P?

   (c) Let γ be a closed path such that none of the roots of P lies on γ. Determine the value of

$$\frac{1}{2\pi i}\int_\gamma (P'/P)(z) dz.$$

Solution:

(a)

$$(fg) \mapsto \frac{(fg)'}{fg} = \frac{f'g}{fg} + \frac{fg'}{fg} = \frac{f'}{f} + \frac{g'}{g}.$$

(b) Denote $\Phi(f) := f'/f$, then by (a),

$$P'/P = \Phi(P(z)) = \sum_{i=1}^n \Phi(z - a_i) = \sum_{i=1}^n \frac{1}{z - a_i}.$$

(c) Applying the result of (b), suppose $a_{k_1}, \cdots, a_{k_m}$ are the set of roots of $P(z)$ lying in γ, then

$$\frac{1}{2\pi i}\int_\gamma \frac{P'}{P} dz = \frac{1}{2\pi i}\sum_{i=1}^m \int_\gamma \frac{1}{z - a_{k_i}} dz = \sum_{i=1}^m (1) = m.$$

$\square$

6. **Find Exercise by Yourself:** Evaluation of integral with the application of Cauchy integral formula and the *keyhole* argument.