MATH4023
Tutorial 7 - Cauchy Integral Formula
April 7/9, 2020 Notes by Ivan So

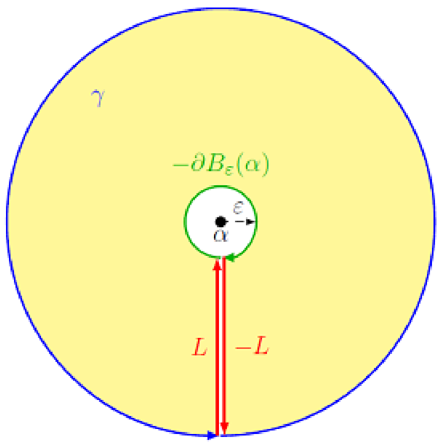
1 Review

Theorem 1.1. Suppose f is *holomorphic* on a open set Ω and contains the closure $\overline{D}$ of the disc D. Then for any $z \in D$,

$$f(z) = \frac{1}{2\pi i}\int_{\partial\overline{D}} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Proof sketch: Consider a *keyhole* domain excluding the point z can apply Cauchy-Goursat's theorem.

- Consider the keyhole domain:



- Applying Cauchy-Goursat's theorem, then

$$\oint_\gamma \frac{f(z)}{z-\alpha}dz = \oint_{|z-\alpha|=\epsilon} \frac{f(z)}{z-\alpha}dz = \boxed{\oint_{|z-\alpha|=\epsilon} \frac{f(z)-f(\alpha)}{z-\alpha}dz} + \oint_{|z-\alpha|=\epsilon} \frac{f(\alpha)}{z-\alpha}dz$$

- The first integral vanishes.
- Second integral can be obtained from direct evaluation, which is $2\pi i f(\alpha)$.

Remark 1.2. The *keyhole* argument of Cauchy integral formula enable us to integrate functions with multiple singularity without the method of *partial fraction*. The idea is closely related to method of **residue calculus** (see later lectures).

2 Problems

1. **True or False**

   (a) The keyhole domain is not simply-connected.

   (b) The reason behind the vanishing of the first term (boxed in the review section) when we apply the Cauchy-Goursat's theorem in the proof of the Cauchy's integral formula is complex differentiability.

2. Let f be holomorphic on $\overline{B}(z_0, b)$. Show that

$$\frac{1}{\pi b^2} \int \int_{\overline{B}_b(z_0)} f(x + iy) dy dx = f(z_0).$$

3. Let f be a holomorphic function on $B_{R_0}(0)$.

   (a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\zeta}) \mathrm{Re}\left(\frac{Re^{i\zeta} + z}{Re^{i\zeta} - z} \right) d\zeta.$$

   (b) Show that

$$\mathrm{Re}\left(\frac{Re^{i\gamma} + r}{Re^{i\gamma} - r} \right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\gamma + r^2}.$$

4. Show that there is no function f holomorphic in $\Omega = \mathbb{C} \setminus [-1, 1]$ such that $f(z)^2 = z$.

5. (a) Show that the association $f \mapsto f'/f$ (for a holomorphic f) sends product to sum.

   (b) If $P(z) = (z - a_1) \cdots (z - a_n)$, what is P'/P?

   (c) Let γ be a closed path such that none of the roots of P lies on γ. Determine the value of

$$\frac{1}{2\pi i} \int_\gamma (P'/P)(z) dz.$$

6. **Find Exercise by Yourself:** Evaluation of integral with the application of Cauchy integral formula and the *keyhole* argument.