MATH4023
Tutorial 5 - Complex Log. and Trig. Functions, Contour Integral
March 24/26, 2020 Notes by Ivan So

1 Review

1.1 Logarithm and Trigonometric Function

Definition 1.1. (complex logarithm) Given $z \in \mathbb{C}$ and $z \neq 0$, the **complex logarithm** of z is the *set*

$$\log z := \{w \in \mathbb{C} : e^w = z\}.$$

Definition 1.2. (principal logarithm) For any $z \neq 0$, the **principal logarithm** is the function

$$\text{Log}(z) := \ln|z| + i\text{Arg}(z).$$

Definition 1.3. (complex sine and cosine) For any $z \in \mathbb{C}$ we define

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Definition 1.4. (inverse trigonometric function) The **inverse trigonometric function** are

$$\sin^{-1} z := \{w \in \mathbb{C} : \sin w = z\}, \qquad \cos^{-1} z := \{w \in \mathbb{C} : \sin w = z\}.$$

1.2 Contour Integral

Definition 1.5. (integral along a curve) Given a *smooth curve* γ parametrize $z : [a,b] \subset \mathbb{R} \to \mathbb{C}$, then the integral of $f(z)$ along γ is defined as

$$\int_\gamma f(z)dz := \int_a^b f(z(t))z'(t)dt.$$

Definition 1.6. (primitive) A **primitive** for a *complex-valued function* f over Ω is a *holomorphic* function F such that $F' = f$ (can think of anti-derivative of a function in 1-variable calculus).

Theorem 1.7. ("fundamental theorem of calculus") If f is a *complex-valued function* with *primitive* in open subset $\Omega \subset \mathbb{C}$ and γ is a curve with initial point and end point respectively w_1, w_2, then

$$\int_\gamma f(z)dz = F(w_2) - F(w_1).$$

Proof sketch: An application of chain rule in line integral. Refer to your knowledge in multivariable calculus.

Lemma 1.8. (integral approximation) The *integration* of *continuous function* over curve γ satisfies the following inequality:

$$\left|\int_\gamma f(z)dz\right| \leq \sup_{z\in\gamma}|f(z)| \cdot \text{length}(\gamma).$$

2 Problems

1. **True or False**

   (a) The function $f(z) = 1/z$ has primitive in $\mathbb{C} \setminus \{0\}$.
   False. Take the path of γ being the closed unit circle, having $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$. Then

   $$\oint_\gamma \frac{1}{z}dz = \int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)}dt = \int_0^{2\pi} idt = 2\pi i \neq 0.$$

   If primitive exists, close-loop integral is zero. Therefore $1/z$ do not have primitive. $\square$

(b) $\mathrm{Log}(z)$ is a multivalued function.
False. Notice that $\mathrm{Arg}(z)$ and $\ln|z|$ are single-valued functions. $\square$

2. Let σ be a vertical segment parametrized by $\sigma(t) = z_0 + itc$, $-1 \le t \le 1$ for $z_0 \in \mathbb{C}$ a fixed complex number, c a fixed real number > 0. Let $\alpha = z_0 + x$ and $\alpha' = z_0 - x$, where x is real positive. Find

$$\lim_{x\to 0} \int_\sigma \left(\frac{1}{z-\alpha} - \frac{1}{z-\alpha'} \right) dz.$$

Solution:

$$\begin{aligned}
\int_\sigma \left(\frac{1}{z-\alpha} - \frac{1}{z-\alpha'} \right) dz &= ic(\alpha - \alpha') \int_{-1}^{1} \frac{dt}{(z-\alpha)(z-\alpha')} \\
&= -\frac{2ix}{c} \int_0^1 \frac{dt}{(x/c)^2 + t^2} \\
&= -4i \arctan\left(\frac{c}{x}\right) \\
\Rightarrow \lim_{x\to 0} \int_\sigma \left(\frac{1}{z-\alpha} - \frac{1}{z-\alpha'} \right) dz &= -2\pi i
\end{aligned}$$

$\square$

3. Let Ω be a simply-connected open set. Let f be a holomorphic function on Ω and assume that $f(z) \neq 0$ for all $z \in \Omega$. Show that there exists holomorphic function g on Ω such that $g^n = f$ for any $n \in \mathbb{N}$.

Solution: Consider the function

$$g_n(z) = e^{\frac{1}{n}\mathrm{Log} f(z)}.$$

One can check easily that it is holomorphic since $\mathrm{Log} f(z) = \ln f(z) + i\mathrm{Arg}(f(z))$. $\mathrm{Log} f(z)$ is holomorphic over Ω since $f(z) \neq 0$. Then the claim easily follows. $\square$

4. (a) Evaluate the integral

$$\int_\gamma z^n dz$$

for all $n \in \mathbb{Z}$, where γ is any circle centered at the origin with the counterclockwise orientation.

(b) Same as (a), but γ is any circle **not** containing the origin.

(c) Show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b},$$

where γ denotes the circle centered at the origin of radius r with counterclockwise orientation.

Solution:

(a) Let R be the radius of the circle,

$$\int_\gamma z^n dz = i \int_0^{2\pi} R^n e^{in\theta} (Re^{i\theta}) d\theta = \begin{cases} 2\pi i & \text{for } n = -1, \\ 0 & \text{for } n \neq -1. \end{cases}$$

(b) Let R be the radius of the circle centered at z_0. By the existence of primitive of z^n for $n \neq -1$, $\int_\gamma z^n dz = 0$. So our only concern will be the case of $n = -1$.
For $n = -1$,

$$\begin{aligned}
\int_\gamma z^n dz &= i \int_0^{2\pi} (z_0 + Re^{i\theta})^n (Re^{i\theta}) d\theta \\
&= \frac{i}{z_0} \int_0^{2\pi} Re^{i\theta} \sum_{n=0}^{\infty} \left(-\frac{R}{z_0} e^{i\theta} \right)^n d\theta \qquad (|z_0| > R) \\
&= \frac{i}{z_0} \sum_{n=0}^{\infty} (-1)^n \frac{R^{n+1}}{z_0^n} \int_0^{2\pi} e^{i(n+1)\theta} d\theta \qquad \text{(LDCT)} \\
&= 0
\end{aligned}$$

(c)

$$
\begin{aligned}
\int_\gamma \frac{1}{(z-a)(z-b)} dz &= \frac{1}{a-b} \int_\gamma \left(\frac{1}{z-a} - \frac{1}{z-b} \right) dz \\
&= \frac{1}{a-b} \int_0^{2\pi} \sum_{n=0}^{\infty} \left((re^{i\theta})^{-n-1} a^n + (re^{i\theta})^n b^{-n-1} \right) ire^{i\theta} d\theta \\
&= \frac{1}{a-b} \sum_{n=0}^{\infty} \int_0^{2\pi} \left((re^{i\theta})^{-n-1} a^n + (re^{i\theta})^n b^{-n-1} \right) ire^{i\theta} d\theta \qquad \text{(LDCT)} \\
&= \frac{2\pi i}{a-b} \qquad \text{(integral except constant term vanishes)}
\end{aligned}
$$

□

5. Prove that $|e^{z_1} - e^{z_2}| \leq |z_1 - z_2|$, where $z_1, z_2 \in \{z : \operatorname{Im}(z) \leq 0\}$.

Solution: Take $\gamma(t) = z_1 + (z_2 - z_1)t,\ t \in [0,1]$ to be the line segment connecting z_1 and z_2, then

$$\int_\gamma e^z dz = e^{z_2} - e^{z_1}.$$

For any $t \in [0,1]$, $|e^{\gamma(t)}| \leq 1$. What remains is just to exploit the theorem of integral approximation:

$$|e^{z_2} - e^{z_1}| = \left| \int_\gamma f(z) dz \right| \leq \sup_{z \in \gamma} |e^z| \cdot |z_2 - z_1|.$$

□