MATH4023
Tutorial 5 - Complex Log. and Trig. Functions, Contour Integral
March 24/26, 2020 Notes by Ivan So

1 Complex Log. and Trig. Functions, Contour Integral

1.1 Logarithm and Trigonometric Function

Definition 1.1. (complex logarithm) Given $z \in \mathbb{C}$ and $z \neq 0$, the **complex logarithm** of z is the *set*

$$\log z := \{w \in \mathbb{C} : e^w = z\}.$$

Definition 1.2. (principal logarithm) For any $z \neq 0$, the **principal logarithm** is the function

$$\mathrm{Log}(z) := \ln |z| + i\mathrm{Arg}(z).$$

Definition 1.3. (complex sine and cosine) For any $z \in \mathbb{C}$ we define

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Definition 1.4. (inverse trigonometric function) The **inverse trigonometric function** are

$$\sin^{-1} z := \{w \in \mathbb{C} : \sin w = z\}, \qquad \cos^{-1} z := \{w \in \mathbb{C} : \sin w = z\}.$$

1.2 Contour Integral

Definition 1.5. (integral along a curve) Given a *smooth curve* γ parametrize $z : [a, b] \subset \mathbb{R} \to \mathbb{C}$, then the integral of $f(z)$ along γ is defined as

$$\int_\gamma f(z)dz := \int_a^b f(z(t))z'(t)dt.$$

Definition 1.6. (primitive) A **primitive** for a *complex-valued function* f over Ω is a *holomorphic* function F such that $F' = f$ (can think of anti-derivative of a function in 1-variable calculus).

Theorem 1.7. ("fundamental theorem of calculus") If f is a *complex-valued function* with *primitive* in open subset $\Omega \subset \mathbb{C}$ and γ is a curve with initial point and end point respectively w_1, w_2, then

$$\int_\gamma f(z)dz = F(w_2) - F(w_1).$$

Proof sketch: An application of chain rule in line integral. Refer to your knowledge in multivariable calculus.

Lemma 1.8. (integral approximation) The *integration* of *continuous function* over curve γ satisfies the following inequality:

$$\left|\int_\gamma f(z)dz\right| \leq \sup_{z\in\gamma} |f(z)| \cdot \mathrm{length}(\gamma).$$

2 Problems

1. **True or False**

   (a) The function $f(z) = 1/z$ has primitive in $\mathbb{C} \setminus \{0\}$.

   (b) $\text{Log}(z)$ is a multivalued function.

2. Let σ be a vertical segment parametrized by $\sigma(t) = z_0 + itc$, $-1 \leq t \leq 1$ for $z_0 \in \mathbb{C}$ a fixed complex number, c a fixed real number > 0. Let $\alpha = z_0 + x$ and $\alpha' = z_0 - x$, where x is real positive. Find

$$\lim_{x \to 0} \int_\sigma \left(\frac{1}{z - \alpha} - \frac{1}{z - \alpha'} \right) dz.$$

3. Let Ω be a simply-connected open set. Let f be a holomorphic function on Ω and assume that $f(z) \neq 0$ for all $z \in \Omega$. Show that there exists holomorphic function g on Ω such that $g^n = f$ for any $n \in \mathbb{N}$.

4. (a) Evaluate the integral

$$\int_\gamma z^n dz$$

for all $n \in \mathbb{Z}$, where γ is any circle centered at the origin with the counterclockwise orientation.

(b) Same as (a), but γ is any circle **not** containing the origin.

(c) Show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b},$$

where γ denotes the circle centered at the origin of radius r with counterclockwise orientation.

5. Prove that $|e^{z_1} - e^{z_2}| \leq |z_1 - z_2|$, where $z_1, z_2 \in \{z : \mathrm{Im}(z) \leq 0\}$.