lecture 09

19/03/2020

Last time

- For a C^1-curve γ parametrized by $z(t) = x(t) + i\,y(t)$, $t \in [a,b]$.

$$\int_\gamma f(z)\,dz := \int_a^b f(z(t))\,\underbrace{z'(t)\,dt}_{dz}$$

- If $\exists\, F : \underset{\substack{\cap \\ \gamma}}{\Omega} \to \mathbb{R}$ s.t. $\frac{d}{dz}F(z) = f(z)$ on Ω

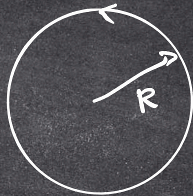
then

$$\int_\gamma f(z)\,dz = F(Q) - F(P).$$

- If $|f(z)| \le M$ on γ, $\text{length}(\gamma) \le L \Rightarrow \left|\int_\gamma f\,dz\right| \le ML$.

e.g.:

$$\lim_{R\to\infty}\oint_{|z|=R}\frac{\operatorname{Log} z}{z^2}\,dz = 0.$$



Sol:

$$\left|\frac{\operatorname{Log} z}{z^2}\right| = \left|\frac{\ln|z| + i\operatorname{Arg} z}{z^2}\right|$$

$$= \frac{|\ln R + i\operatorname{Arg} z|}{R^2} \quad \text{on } |z|=R.$$

$$\le \frac{\ln R + |\operatorname{Arg} z|}{R^2} \le \overbrace{\frac{\ln R + \pi}{R^2}}^{M}$$

$$\left|\oint_{|z|=R}\frac{\operatorname{Log} z}{z^2}\,dz\right| \le \frac{\ln R+\pi}{R^2}\cdot \underbrace{2\pi R}_{\text{length } \bigcirc} = \frac{2\pi(\ln R+\pi)}{R} \longrightarrow 0 \quad \text{as } R\to\infty.$$

$$\Rightarrow \lim_{R\to\infty}\oint_{|z|=R}\frac{\operatorname{Log} z}{z^2}\,dz = 0.$$

Proof?

$$\int_\gamma f(z)\,dz = \int_a^b (u+iv)\,(x'+iy')\,dt \qquad f = u+iv.$$

$$dz = z'(t)\,dt = (x'+iy')\,dt.$$

$$= \int_a^b (ux'-vy') + i\,(uy'+vx')\,dt$$

$$z(t) = x(t) + i\,y(t)$$

$$\int u\,dx - v\,dy \qquad \int u\,dy + v\,dx$$

$$\left|\int_\gamma f(z)\,dz\right| = \sqrt{\Big(\underbrace{\int_a^b (ux'-vy')\,dt}_{\le M}\Big)^2 + \Big(\underbrace{\int_a^b uy'+vx'\,dt}_{\le ML}\Big)^2} \le \sqrt{2}\,ML.$$

(Cauchy-Schwarz)

$$|ux'-vy'| \le \underbrace{\sqrt{u^2+v^2}}_{=\,|f(z)|\,\le\, M} \cdot \sqrt{(x')^2+(y')^2}$$

$$\le M\sqrt{(x')^2+(y')^2}$$

$$\int_a^b (ux'-vy')\,dt \le \int_a^b M\sqrt{(x')^2+(y')^2}\,dt \le ML.$$

Proof of ML-inequality:

$$I = \int_\gamma f(z)\,dz \qquad \text{Write} \quad I = |I|\,e^{i\theta}.$$

$$\Rightarrow \underset{\in\mathbb{R}}{|I|} = \underbrace{I e^{-i\theta}}_{\in\mathbb{R}}.$$

$$\underset{\in\mathbb{R}}{I e^{-i\theta}} = \int_\gamma e^{-i\theta} f(z)\,dz$$

$$= \int_a^b \left(\mathrm{Re}\left(e^{-i\theta} f(z)\right) + i\,\mathrm{Im}\left(e^{-i\theta} f(z)\right)\right)\cdot \underbrace{(x' + iy')\,dt}_{dz}$$

$$= \int_a^b \left(\mathrm{Re}\left(e^{-i\theta} f(z)\right) x' - \mathrm{Im}\left(e^{-i\theta} f(z)\right) y'\right) dt \;\; \underbrace{+\, i\left(- \;\; - \;\; - \;\; - \;\; - \right)}_{=0}$$

$$|I| = |I e^{-i\theta}| \le \int_a^b \left|\mathrm{Re}\left(e^{-i\theta} f\right) x' - \mathrm{Im}\left(e^{-i\theta} f\right) y'\right| dt$$

$$\le \int_a^b \sqrt{\mathrm{Re}\left(e^{-i\theta} f\right)^2 + \mathrm{Im}\left(e^{-i\theta} f\right)^2} \cdot \sqrt{(x')^2 + (y')^2}\,dt$$

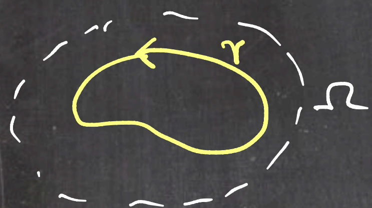
$$= \int_a^b \underbrace{\left|e^{-i\theta} f(z)\right|}_{= |f(z)| \le M} \cdot \sqrt{(x')^2 + (y')^2}\,dt$$

$$\le M \int_a^b \sqrt{(x')^2 + (y')^2}\,dt \le ML.$$

Cauchy-Goursat's Theorem.

Simple closed curve $\gamma \subset \Omega$

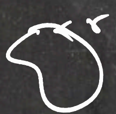
simply-connected domain

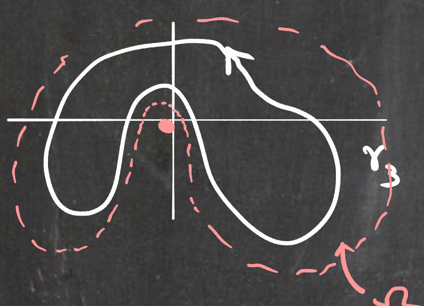


f is holomorphic on Ω. $\overset{\text{then}}{\Longrightarrow}$ $\oint_\gamma f(z)\,dz = 0$.

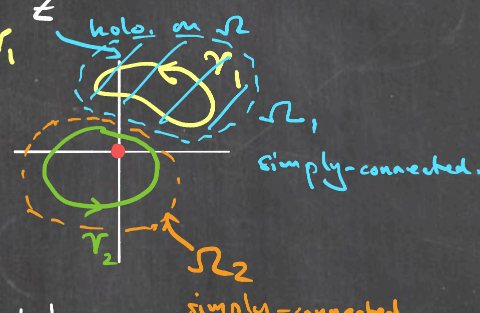
$\oint_\gamma e^z\,dz = 0$.

holo. on $\mathbb{C}$.





$\oint_{\gamma_1} \frac{1}{z}\,dz = 0$.

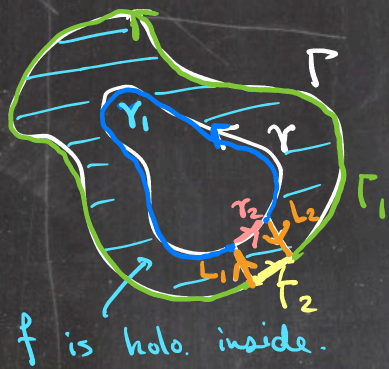


$\oint_{\gamma_2} \frac{1}{z}\,dz$

Cauchy-Goursat does not apply.

$\oint_{\gamma_3} \frac{1}{z}\,dz = 0$.

e.g.



Claim:

$$\oint_\Gamma f(z)\,dz = \oint_\gamma f(z)\,dz .$$

⊠

$$\int_{\Gamma_1} f(z)\,dz + \int_{L_1} f(z)\,dz - \int_{\gamma_1} f(z)\,dz + \int_{L_2} f(z)\,dz = \oint f(z)\,dz \underset{\text{C.G.}}{=} 0$$

$+$

$$\int_{\Gamma_2} f(z)\,dz - \int_{L_2} f(z)\,dz - \int_{\gamma_2} f(z)\,dz - \int_{L_1} f(z) = \oint f(z)\,dz \overset{\text{C.G.}}{=} 0.$$

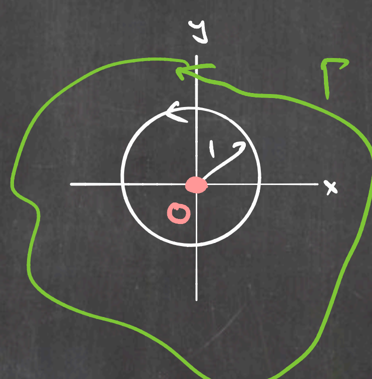
$$\oint_\Gamma f(z)\,dz \quad - \oint_\gamma f(z)\,dz \quad = 0.$$

⊠

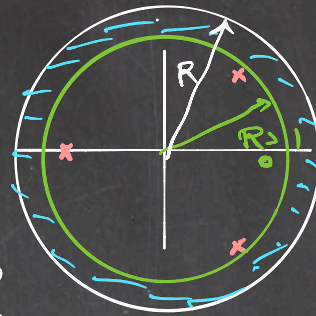
$$\oint_{|z|=1} \frac{1}{z}\,dz = 2\pi i$$

$$\| $$

$$\oint_\Gamma \frac{1}{z}\,dz$$



$$\oint_{|z|=R} \frac{z-1}{z^3+1}\,dz = ?$$



$$\left| \oint_{|z|=R} \underbrace{\frac{z-1}{z^3+1}}\,dz \right| \le \frac{(R+1)(2\pi R)}{R^3-1} \to 0$$

as $R \to \infty$.

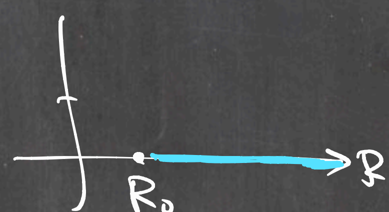
$$\frac{R+1}{R^3-1}$$

$$|z-1| \le |z| + |-1| = |z| + 1.$$

$$|z^3+1| \ge |z|^3 - 1 = R^3 - 1.$$

$$\underbrace{\lim_{R\to\infty} \oint_{|z|=R} \frac{z-1}{z^3+1}\,dz} = 0.$$

$$\oint_{|z|=R} \frac{z-1}{z^3+1}\,dz = \oint_{|z|=R_0} \frac{z-1}{z^3+1}\,dz \qquad \forall R \ge R_0.$$



$$|z+w| \le |z| + |w|$$

$$|z \pm w| \ge \big|\,|z| - |w|\,\big|.$$