香港科技大學
THE HONG KONG
UNIVERSITY OF SCIENCE
AND TECHNOLOGY

數學系
DEPARTMENT OF MATHEMATICS

# MIDTERM EXAMINATION

**Course Code:** MATH 4023
**Course Title:** Complex Analysis
**Semester:** Spring 2016-17
**Date and Time:** 16 March 2017, 7:00PM - 9:00PM

## Instructions

- Do **NOT** open the exam until instructed to do so.
- All mobile phones and communication devices should be switched **OFF**.
- Use of calculators is **NOT** allowed.
- It is a **CLOSED-BOOK** and **CLOSED-NOTES** exam.
- Answer **ALL** problems. Write your answers in Part A in the spaces provided, and write your solutions to problems in Part B in the yellow book.
- You must **SHOW YOUR WORK** to receive credits in all problems in Part B.
- Some problems are structured into several parts. You can quote the results stated in the preceding parts to do the next part.

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## HKUST Academic Honor Code

Honesty and integrity are central to the academic work of HKUST. Students of the University must observe and uphold the highest standards of academic integrity and honesty in all the work they do throughout their program of study. As members of the University community, students have the responsibility to help maintain the academic reputation of HKUST in its academic endeavors. Sanctions will be imposed on students, if they are found to have violated the regulations governing academic integrity and honesty.

"I confirm that I have answered the questions using only materials specified approved for use in this examination, that all the answers are my own work, and that I have not received any assistance during the examination."

**Student's Signature:** ______________________________

**Student's Name:** ______________________ **HKUST ID:** ____________

## Part A - Short Questions (40 points)

1. Which ONE of the following is different from the other four? Put ✓ in the correct answer: **/2**
   - ◯ $\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}$
   - ◯ $\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)^2$
   - ◯ $\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)^{\frac{1}{3}}$
   - ◯ $e^{\overline{\frac{\pi}{6}i}}$
   - ◯ $\overline{e^{\frac{\pi}{6}i}}$

2. Write down the set of all complex number(s) $z$ such that the series $\sum_{n=1}^{\infty} \frac{e^{in}z^n}{n}$ converges absolutely. **/3**

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3. Consider the following subset $\Omega$ in $\mathbb{C}$:

$$\Omega = \{z : |z+1| < 1\} \cup \{z : \mathrm{Re}(z) \geq 0\}.$$

Answer the following questions. No justification is needed.

(a) Is 0 an interior, exterior or boundary point of $\Omega$? Please circle: **/2**

interior    exterior    boundary

none of the above

(b) Which of the following terms correctly describe $\Omega$? Circle ALL correct answer(s): **/4**

open    not open    closed    not closed

connected    disconnected    simply-connected    not simply-connected

bounded    unbounded    compact    non-compact

4. Suppose $E$ is a non-empty closed set in $\mathbb{C}$. Which of the following must be true? Put ✓ in ALL correct answer(s): **/4**
   - ◯ Every sequence $\{z_n\}_{n=1}^{\infty}$ in $E$ has a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ that converges to a limit $w \in E$ as $k \to \infty$.
   - ◯ Every convergent sequence $\{z_n\}_{n=1}^{\infty}$ in $E$ has its limit $w$ in $E$.
   - ◯ $\partial E \subset E$
   - ◯ $\mathbb{C} \backslash E$ is closed.

5. Suppose $f(x+yi) = u(x,y) + iv(x,y)$ is complex differentiable at $z_0 = x_0 + y_0 i$. Which of the following MUST be true? Put ✓ in ALL correct answer(s): **/4**

   ◯ $u$ and $v$ are real differentiable at $(x_0, y_0)$

   ◯ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at $(x_0, y_0)$

   ◯ $\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x}$ at $(x_0, y_0)$

   ◯ $f$ is holomorphic on $B_\varepsilon(z_0)$ for some $\varepsilon > 0$.

6. Suppose $f(x+yi) = u(x,y) + iv(x,y)$ satisfies: **/4**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{at } (x_0, y_0).$$

   Which of the following MUST be true? Put ✓ in ALL correct answer(s):

   ◯ $u$ and $v$ are real differentiable at $(x_0, y_0)$

   ◯ $f$ is complex differentiable at $z_0 = x_0 + y_0 i$

   ◯ $\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x}$ at $(x_0, y_0)$

   ◯ $f$ is holomorphic on $B_\varepsilon(z_0)$ for some $\varepsilon > 0$.

7. Which of the following functions is/are holomorphic on $\mathbb{C}\backslash\{0\}$? Put ✓ in ALL correct answer(s): **/2**

   ◯ $f(z) = \mathrm{Log}(z)$

   ◯ $f(z) = 1/\overline{z}$

   ◯ $f(z) = \sum_{n=0}^{\infty} \frac{1}{n! z^n}$

   ◯ $f(z) = \frac{1}{1+z^4}$

8. Let $\Omega$ be an open subset of $\mathbb{C}$. Suppose there are two functions $F : \Omega \to \mathbb{C}$ and $f : \Omega \to \mathbb{C}$ such that $F'(z) = f(z)$ on $\Omega$. Which of the following must be true? Put ✓ in ALL correct answer(s): **/4**

   ◯ $\oint_\gamma \overline{f(z)}\, dz = 0$ for any closed $C^1$ curve in $\Omega$.

   ◯ $\oint_\gamma f(z)\, dz = 0$ for any closed $C^1$ curve in $\Omega$.

   ◯ $\oint_\gamma F(z)\, dz = 0$ for any closed $C^1$ curve in $\Omega$.

   ◯ $\oint_\gamma f(z)F(z)\, dz = 0$ for any closed $C^1$ curve in $\Omega$.

9. State the Cauchy's integral formula by filling in the box below: **/3**

   Suppose the function $f : \Omega \to \mathbb{C}$ is holomorphic on a simply-connected domain $\Omega$. Let $\gamma$ be a simple closed curve in $\Omega$, and $\alpha$ be a point enclosed by $\gamma$. Then, we have:

$$\oint_\gamma \frac{f(z)}{z-\alpha}\,dz = \boxed{\phantom{xxxxxxxxxxxx}}$$

10. Answer the following questions about the proof of Cauchy-Goursat's Theorem (step 1). As a reference, an extract of the proof is included in this exam.

    (a) In one step, we defined $E(z) := f(z) - f(z_0) - f'(z_0)(z - z_0)$ where $z_0$ is a point at which $f$ is complex differentiable. Then, we claimed that **/4**

$$\oint_T E(z)\,dz = \oint_T f(z)\,dz$$

    where $T$ is a closed triangle. Explain why the two integrals are equal in the proof.

    (b) Briefly explain why the proof would fail if the domain $\Omega$ were not simply-connected. **/4**

## Part B - Long Questions (60 points)

*Instructions:*

- *Write your solutions in the yellow book provided. Clearly indicate the problem and part numbers. Open a new page for each problem.*
- *Problems are not necessarily arranged by the level of difficulty.*
- *Unless otherwise is stated, $x$, $y$, $u$ and $v$ are assumed to be real.*

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1. Find ALL entire functions $f : \mathbb{C} \to \mathbb{C}$ in the form of: **/10**

$$f(x+yi) = u(x,y) + iv(y).$$

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2. (a) Let $a > 0$. Given that $f$ is holomorphic on $\Omega := \{z \in \mathbb{C} : |z| > a\}$, $\gamma$ is a simple closed curve in $\Omega$, and $\gamma$ is contained inside the ball $B_R(0)$ (where $R > a$). Using Cauchy-Goursat's Theorem, show that: **/20**

$$\oint_\gamma f(z)\,dz = \oint_{|z|=R} f(z)\,dz.$$

[Remark: We did a similar proof in class. However, please show your work, and do not just say something like "from class, the two integrals are equal".]

(b) Let $N$ be your HKUST student ID, and $p(z)$ be a polynomial of degree at most 4023, and $\Gamma$ be a simple closed curve enclosing all points $0, 1, 2, \ldots, N$ in $\mathbb{C}$. Using (a), or otherwise, show that:

$$\oint_\Gamma \left( \frac{1}{2017z} + 2z^{2016} + \frac{p(z)}{(z-1)(z-2)\cdots(z-N)} \right) dz = \frac{2\pi i}{2017}.$$

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3. (a) Show that for any integer $N \geq 2$ and $z \in \mathbb{C}\backslash\{1\}$, we have: **/15**

$$1 + 2z + 3z^2 + \cdots + Nz^{N-1} = \frac{1-(N+1)z^N}{1-z} + \frac{z(1-z^N)}{(1-z)^2}.$$

(b) Given that $\{a(n,k)\}_{n,k=0}^{\infty}$ is a double-index sequence of complex numbers such that:

$$\sum_{n=k}^{2016} a(n,k) = k+1 \qquad \text{for any } k \in \{0, 1, \ldots, 2016\}.$$

Using (a), show that:

$$\sum_{n=1}^{2016} \sum_{k=1}^{n} a(n,k) \sin\frac{2k\pi}{2017} = -\frac{2017}{2} \cot\frac{\pi}{2017}.$$

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4. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f(z) \neq 0$ for any $z \in \mathbb{C}$. Suppose $f'(z)$ is also an entire function. Show that there exists an **entire** function $g : \mathbb{C} \to \mathbb{C}$ such that $f(z) = e^{g(z)}$ for any $z \in \mathbb{C}$. [Hint: consider the function: **/15**

$$h(z) = \int_{L(0,z)} \frac{f'(\xi)}{f(\xi)}\,d\xi$$

where $L(0,z)$ is the straight-path from 0 to $z$.]

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End of Exam