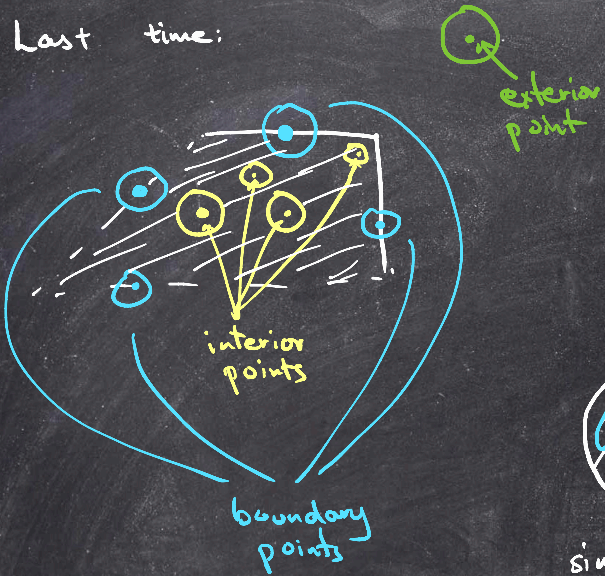
Lecture 05

05/03/2020

Last time:



open, not closed

$\Omega^\circ = \Omega$

$\square = \partial\Omega \not\subset \Omega$

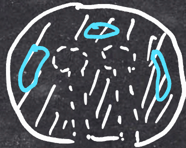
not open, $\Omega^\circ = \square$

not closed $\underbrace{\square}_{\partial\Omega} \not\subset \Omega$

not open, closed. $\partial\Omega = \square$

$\cap$

$\blacksquare$



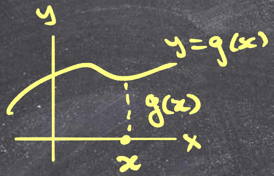
simply-connected

not simply-con.

Ch. 2 $\quad f : \Omega \subset \mathbb{C} \to \mathbb{C} \qquad \boxed{w = f(z)}$

$$u + iv = f(x+yi)$$

$y = g(x)$



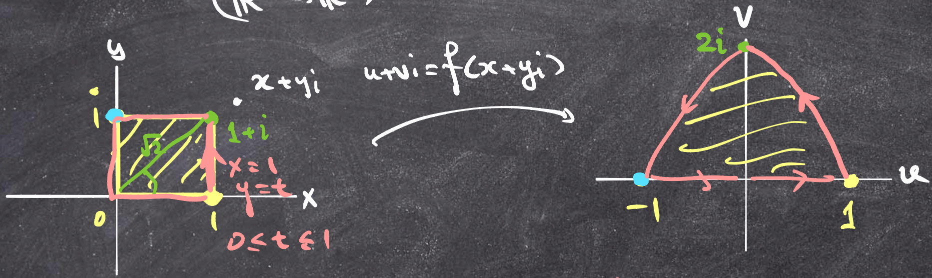
$$u(x+iy) : \mathbb{R}^2 = \mathbb{C} \to \mathbb{R}$$
$$v(x+iy) : \mathbb{C} \to \mathbb{R}$$

e.g. $\quad f(z) = z^2$.

$$f(x+yi) = (x+yi)^2 = \underbrace{(x^2-y^2)}_{u(x,y)} + \underbrace{2xy}_{v(x,y)}\, i$$

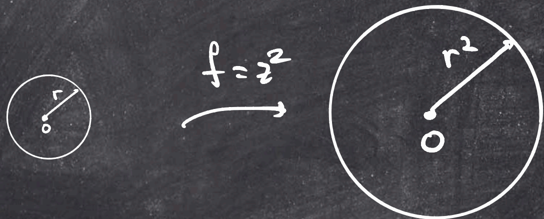
level sets of u: $\quad \{(x,y) : u(x,y) = c\}$.

$$f(z) = z^2 : \mathbb{C} \to \mathbb{C}.$$
$$(\mathbb{R}^2 \to \mathbb{R}^2)$$



$$f(1+ti) = (1+ti)^2 = \underbrace{(1-t^2)}_{u} + \underbrace{2t}_{v}i.$$

$$\left.\begin{aligned} u &= 1-t^2 \\ v &= 2t \Rightarrow t = \frac{v}{2} \end{aligned}\right\} \Rightarrow \boxed{u = 1 - \frac{v^2}{4}}$$

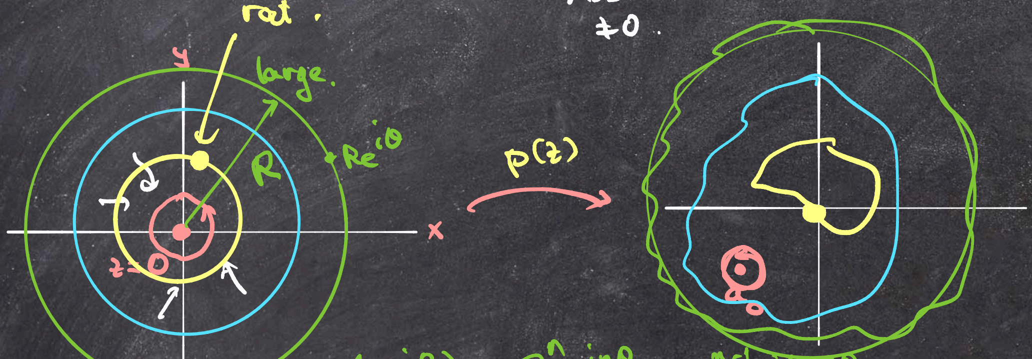


Fundamental Theorem of Algebra:

"Every complex polynomial (non-constant) must have at least one complex root."

$$p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + \underbrace{a_0}_{\text{Assume } \neq 0}$$

(trivial if $a_0 = 0$)



$$p(Re^{i\theta}) = \underbrace{R^n e^{in\theta}}_{\text{largest!}} + R^{n-1}e^{i(n-1)\theta} + \dots + a_0 .$$

$f : \Omega \subset \mathbb{C} \to \mathbb{C}$.

Ω: open connected

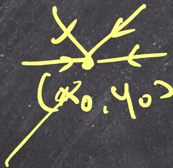
Def: $z_0 \in \Omega$, f is complex differentiable at z_0

$z_0 = x_0 + y_0 i$

$\overset{def}{\iff}$ $\displaystyle\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists (if exists then $f'(z_0) := \displaystyle\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$)

$z \to z_0$: $(x,y) \to (x_0, y_0)$

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{w \to 0} \frac{f(z_0+w)-f(z_0)}{w}$$



e.g. $f(z) = z^n \quad (n \in \mathbb{N})$

$$\lim_{w \to 0} \frac{f(z+w) - f(z)}{w} = \lim_{w \to 0} \frac{(z+w)^n - z^n}{w}$$

$$= \lim_{w \to 0} \frac{\cancel{z^n} + \binom{n}{1} z^{n-1} w + \binom{n}{2} z^{n-2} w^2 + \dots + w^n - \cancel{z^n}}{w}$$

$$= \lim_{w \to 0} \underline{n z^{n-1}} + \underline{\binom{n}{2} z^{n-2} w + \dots + w^{n-1}}$$

$$= n z^{n-1}.$$

f is complex diff. at any $z \in \mathbb{C}$.

$f'(z) = n z^{n-1}$.

e.g. $f(z) = \bar{z}$.

$$\lim_{w \to 0} \frac{f(z+w) - f(z)}{w} = \lim_{w \to 0} \frac{\overline{\cancel{z}+w} - \cancel{\bar{z}}}{w} = \lim_{w \to 0} \frac{\bar{w}}{w}.$$

$w = h + ki$

real

0 $\quad h \to 0^+$, $k = 0$

$$\lim_{\substack{h \to 0^+ \\ k=0}} \frac{\overline{h+ki}}{h+ki} = \lim_{h \to 0^+} \frac{h}{h} = 1.$$

0 $\quad h = 0$, $k \to 0^+$

$$\lim_{\substack{k \to 0^+ \\ h=0}} \frac{\overline{h+ki}}{h+ki} = \lim_{k \to 0^+} \frac{-ki}{ki}$$

$$= \lim_{k \to 0^+} (-1) = -1$$

$f(z) = \bar{z}$ is nowhere complex diff. on $\mathbb{C}$.

If $f : \Omega \to \mathbb{C}$ is complex differentiable at $z_0 \in \Omega$.

$$z_0 = x_0 + y_0 i$$

then:
$$\underbrace{\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases}}_{\text{Cauchy-Riemann's equations.}} \quad \text{at } (x_0, y_0).$$

$$u + iv = f(x + yi)$$

$$w = h + ki$$

Proof: $f'(z_0) = \lim\limits_{w \to 0} \dfrac{f(z_0 + w) - f(z_0)}{w}$ exists.

$$\lim_{\substack{h \to 0 \\ k = 0}} \frac{(u+iv)(x_0 + iy_0 + h + \cancel{ki}) - (u+iv)(x_0, y_0)}{h + \cancel{ki}}$$

$$= \lim_{h \to 0} \frac{\big(u(x_0 + h + iy_0) - u(x_0, y_0)\big) + i\big(v(x_0 + iy_0 + h) - v(x_0, y_0)\big)}{h}$$

$$= \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} + i \left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}.$$

$$\lim_{\substack{k \to 0 \\ h = 0}} \frac{(u+iv)(x_0, y_0+k) - (u+iv)(x_0, y_0)}{\cancel{h} + ik}$$

$$= \frac{1}{i}\left(\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)} + i \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)} \right)$$

$$= \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)} - i \left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)} \qquad \square$$