MATH4023
Tutorial 1 - Complex Numbers, Polar Form
February 25/27, 2020 Notes by Ivan So

1 Review

Definition 1.1. (complex numbers) **Complex numbers** is the *ring* (field) $\mathbb{C} = \{z = a + bi : a, b \in \mathbb{R}, i^2 = -1\}$.

- a is the *real part* of z, denoted by $\text{Re}\,(z)$.
- b is the *imaginary part* of z, denoted by $\text{Im}(z)$.

Definition 1.2. (conjugate and modulus) Given $z = a + bi \in \mathbb{C}$, we denote and define:

- $\bar{z} := a - bi$ as the **conjugate** of z;
- $|z| := \sqrt{a^2 + b^2}$ as the **modulus** of z.

Remark 1.3. Useful identities: $\bar{z}z = |z|^2$, $\bar{\bar{z}} = z$, $|\bar{z}| = |z|$, $\text{Re}\,(z) = \frac{z+\bar{z}}{2}$, $\text{Im}(z) = \frac{z-\bar{z}}{2i}$, $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$, $\overline{z_1 z_2} = \overline{z_1}\;\overline{z_2}$, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$.

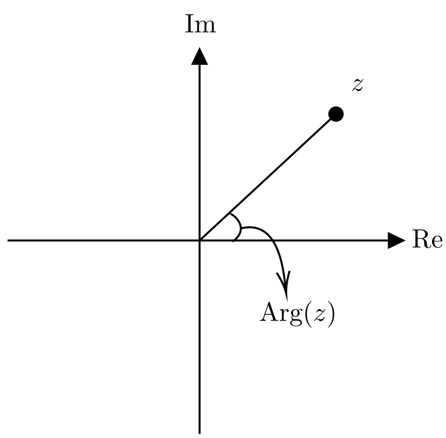
Proposition 1.4. (triangle inequality) Let $z_1, z_2 \in \mathbb{C}$ we have

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Definition 1.5. (polar form and principal argument) Given a complex z, it's **polar form** is defined to be

$$z = |z|(\cos\theta_0 + i\sin\theta_0).$$

The **principal argument** denoted by $\text{Arg}(z)$ is defined to be the angle $\theta_0 \in (-\pi, \pi]$ representing the angle between the origin-z line and the real axis.



Definition 1.6. (argument map) The **argument map** is the map defined on $\mathbb{C} \setminus \{0\}$ by

$$\arg(z) = \{\text{Arg}(z) + 2k\pi i : k \in \mathbb{Z}\}.$$

Proposition 1.7. (De Moivre's Theorem) For any $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$, we have

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Proof sketch: Induction and product to sum formula.

Definition 1.8. (roots of complex number) Given any $z \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$, the n-th **roots** of z is given by

$$z^{\frac{1}{n}} := \left\{ \sqrt[n]{|z|} \left(\cos\left(\frac{\text{Arg}(z) + 2k\pi}{n}\right) + i\sin\left(\frac{\text{Arg}(z) + 2k\pi}{n}\right)\right),\ k \in \{1, \cdots, n-1\}\right\}.$$

Remark 1.9. Notice that $z^{\frac{1}{n}}$ is **multivalued** and different from $\sqrt[n]{z}$.

2 Problems

1. **True or False**

   (a) For any $\theta \in \mathbb{R}$ and $q \in \mathbb{Q}$, we have $(\cos\theta + i\sin\theta)^q = \cos(q\theta) + i\sin(q\theta)$.
   False. For any $q \in \mathbb{Q}$, $q = p/r$ for $p \in \mathbb{Z}$ and $r \in \mathbb{Z} \setminus \{0\}$. Suppose $r \neq 1$, then $(\cos\theta + i\sin\theta)^q$ is *multivalued.* $\square$

   (b) Suppose $a \in \mathbb{R}$, $\sqrt[n]{a} = a^{\frac{1}{n}}$.
   False. $\sqrt[n]{a}$ is denoting the n-th root of a in the real number system, which is single valued. While $a^{\frac{1}{n}}$ is a set of n values. $\square$

2. (a) Let $z = \cos\theta + i\sin\theta$, where $\theta \in \mathbb{R}$. Find the four values of z such that $\operatorname{Im}(z^2 + \overline{z}) = 0$.

   (b) Let z_1, z_2 be two values of z obtained in (a) such that $\operatorname{Im}(z_1) < 0 < \operatorname{Im}(z_2)$. For any positive n, define $S_n = \sum_{r=1}^{n} \omega^r$, where $\omega = \frac{z_2}{z_1}$.

   (i) Prove that $\omega^3 = 1$.

   (ii) If n is a multiple of 3, prove that $S_n = 0$.

   (iii) Does there exist an integer m such that $(S_{2009} + S_{2010} + S_{2011})^m = 2$? Explain.

   (iv) Find all positive integers k such that $(S_n)^k + (S_{n+1})^k + (S_{n+2})^k = 2$ for any positive integer n.

Solution:

(a) From De Moivre's theorem, $z^2 + \overline{z} = (\cos 2\theta + i\sin 2\theta) + (\cos\theta - i\sin\theta)$. So $\operatorname{Im}(z^2 + \overline{z}) = 0$ implies

$$\begin{aligned} \sin 2\theta - \sin\theta &= 0 \\ (2\cos\theta - 1)\sin\theta &= 0 \\ &\Rightarrow \theta = 0 \text{ or } \pi \text{ or } \frac{\pi}{3} \text{ or } \frac{5}{3}\pi. \end{aligned}$$

So the corresponding z's are

$$z_1 = 1 \text{ and } z_2 = -1 \text{ and } z_3 = \frac{1}{2} + \frac{\sqrt{3}}{2}i \text{ and } z_4 = \frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

(b) (i) From our notation in (a),

$$\begin{aligned} \omega = \frac{z_3}{z_4} &= \frac{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}{\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}} = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} \qquad \text{(De Moivre's)} \\ \Rightarrow \omega^3 &= \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3 = \cos\frac{6\pi}{3} + i\sin\frac{6\pi}{3} \qquad \text{(De Moivre's)} \end{aligned}$$

(ii) Let $n = 3k$. Since $\omega^3 = 1$,

$$S_n = k(\omega + \omega^2 + \omega^3).$$

From the geometric sum formula,

$$k(\omega + \omega^2 + \omega^3) = k\omega\frac{1 - \omega^3}{1 - \omega} = 0 \qquad (\omega^3 = 1).$$

(iii) Notice that 2010 divide 3. By (b)(ii),

$$(S_{2009} + S_{2010} + S_{2011})^m = (\omega + (\omega + \omega^2))^m = (\omega - 1)^m.$$

$|\omega - 1| = \sqrt{\left(\cos\frac{2\pi}{3} - 1\right)^2 + \sin^2\frac{2\pi}{3}} = \sqrt{2 - 2\cos\frac{2\pi}{3}} = \sqrt{3}$. For any positive integer m, $(\sqrt{3})^m$ cannot be equal to m. This disproved the claim.

(iv) For 3 consecutive integers, one of them must be divisible by 3. Without loss of generality, assume $n+2$ is divisible by 3. So

$$\begin{aligned} (S_n)^k + (S_{n+1})^k + (S_{n+2})^k &= (\omega)^k + (\omega + \omega^2 + 1 - 1)^k + 0 \\ &= \omega^k + (-1)^k \end{aligned}$$

$|\omega| = 1$ implies $\omega^k + (-1)^k = 2$ if and only if k is even and $\omega^k = 1$. The least positive integer for which $\omega^k = 1$ is 3. Therefore, the set of all k in which $(S_n)^k + (S_{n+1})^k + (S_{n+2})^k = 2$ will be $6\mathbb{Z}$.

□

3. Prove that

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(n+1/2)\theta]}{2\sin(\theta/2)}.$$

Solution: From de Moivre's formula,

$$\begin{aligned}(\cos\theta + i\sin\theta) + \cdots + (\cos\theta + i\sin\theta)^n &= (\cos\theta + i\sin\theta) + \cdots + (\cos n\theta + i\sin n\theta)\\ \frac{1-(\cos\theta+i\sin\theta)^{n+1}}{1-(\cos\theta+i\sin\theta)} &= (\cos\theta + \cdots + \cos n\theta) + i(\sin\theta + \cdots + \sin n\theta)\end{aligned}$$

Therefore,

$$\begin{aligned}\cos\theta + \cdots + \cos n\theta &= \mathrm{Re}\left(\frac{1-(\cos\theta+i\sin\theta)^{n+1}}{1-(\cos\theta+i\sin\theta)}\right)\\ &= \frac{1}{2}\left(\frac{1-(\cos(n+1)\theta + i\sin(n+1)\theta)}{1-(\cos\theta+i\sin\theta)} + \frac{1-(\cos(n+1)\theta - i\sin(n+1)\theta)}{1-(\cos\theta-i\sin\theta)}\right)\\ &= \frac{1+\cos(n+1)\theta\cos\theta + \sin(n+1)\theta\sin\theta - \cos\theta - \cos(n+1)\theta}{2(1-\cos\theta)}\\ &= \frac{1+\cos n\theta - \cos\theta - \cos(n+1)\theta}{4\sin^2(\theta/2)} \quad \text{(compund angle formula + half angle formula)}\\ &= \frac{2\cos^2[(n/2)\theta] - 2\cos[(n/2+1)\theta]\cos[(n/2)\theta]}{4\sin^2(\theta/2)} \quad \text{(half anle formula + product to sum formula)}\\ &= \frac{\cos[(n/2)\theta](\sin[(n/2+1/2)\theta])}{2\sin(\theta/2)} \quad \text{(sum to product formula)}\\ &= \frac{1}{2} + \frac{\sin[(n+1/2)\theta]}{2\sin(\theta/2)} \quad \text{(product to sum formula)}\end{aligned}$$

□

4. Suppose P is a polynomial with real coefficients. Show that $P(z_0) = 0$ **iff** $P(\overline{z_0}) = 0$.

Solution: Let $P(z) = a_d z^d + a_{d-1}z^{d-1} + \cdots + a_0$ with $a_i \in \mathbb{R}$. If $P(z_0) = a_d z_0^d + a_{d-1}z_0^{d-1} + \cdots + a_0 = 0$, then

$$\overline{P(z_0)} = \overline{a_d z_0^d + a_{d-1}z_0^{d-1} + \cdots + a_0} = a_d\overline{z_0}^d + a_{d-1}\overline{z_0}^{d-1} + \cdots + a_0 = P(\overline{z_0}) = \overline{0} = 0$$

This showed $\overline{z_0}$ is also a root of $P(z)$. The prove in the other direction is similar.
Significance: Complex roots for a polynomial comes in a conjugate pair. So, anything special regarding roots of *odd* degree polynomial based on this property? □

5. (a) Show that the n-th roots of 1 (other than 1 itself) satisfy the **cyclotomic equation**

$$z^{n-1} + z^{n-2} + \cdots + z + 1 = 0.$$

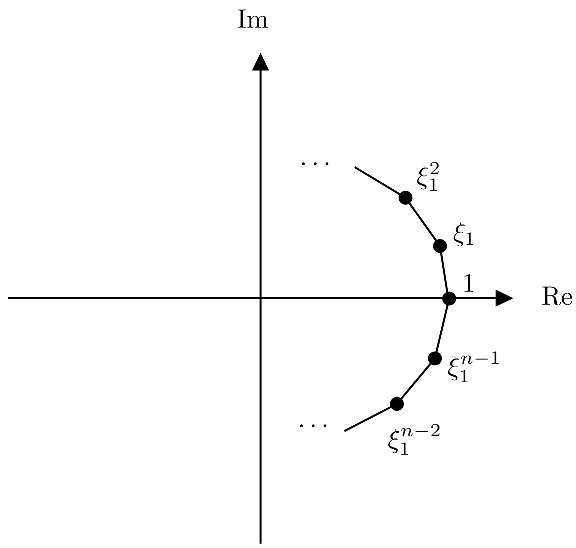
(b) Suppose we consider the $n-1$ diagonals of a regular n-gon inscribed in a unit circle obtained by connecting one vertex with all the others. Show that the product of their lengths is n.

Solution:

(a) Suppose ξ is a n-th root of 1 not equal to 1, then

$$\xi^{n-1} + \xi^{n-2} + \cdots + \xi + 1 = \frac{1-\xi^n}{1-\xi} = \frac{1-1}{1-\xi} = 0.$$

(b) Represent each vertex of a regular n-gon by an element of $1^{\frac{1}{n}}$:



Our target will be to evaluate $|\xi_1 - 1| \cdots |\xi_1^{n-1} - 1|$. Notice that

$$\begin{aligned} z^n - 1 &= (z - \xi_1) \cdots (z - \xi_1^{n-1})(z - 1) \\ \Rightarrow (z - \xi_1) \cdots (z - \xi_1^{n-1}) &= z^{n-1} + \cdots + z + 1 \\ \Rightarrow |(1 - \xi_1) \cdots (1 - \xi_1^{n-1})| &= 1 + \cdots + 1 = n \end{aligned}$$

This proved the claim.

$\square$