MATH4023
Tutorial 1 - Complex Numbers, Polar Form
February 25/27, 2020 Notes by Ivan So

1 Review

Definition 1.1. (complex numbers) **Complex numbers** is the *ring* (field) $\mathbb{C} = \{z = a + bi : a, b \in \mathbb{R}, i^2 = -1\}$.

- a is the *real part* of z, denoted by $\mathrm{Re}\,(z)$.
- b is the *imaginary part* of z, denoted by $\mathrm{Im}(z)$.

Definition 1.2. (conjugate and modulus) Given $z = a + bi \in \mathbb{C}$, we denote and define:

- $\bar{z} := a - bi$ as the **conjugate** of z;
- $|z| := \sqrt{a^2 + b^2}$ as the **modulus** of z.

Remark 1.3. Useful identities: $\bar{z}z = |z|^2$, $\bar{\bar{z}} = z$, $|\bar{z}| = |z|$, $\mathrm{Re}\,(z) = \frac{z+\bar{z}}{2}$, $\mathrm{Im}(z) = \frac{z-\bar{z}}{2i}$, $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$, $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$.

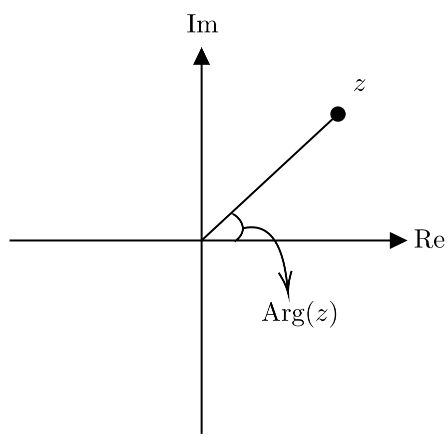
Proposition 1.4. (triangle inequality) Let $z_1, z_2 \in \mathbb{C}$ we have

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Definition 1.5. (polar form and principal argument) Given a complex z, it's **polar form** is defined to be

$$z = |z|(\cos\theta_0 + i\sin\theta_0).$$

The **principal argument** denoted by $\mathrm{Arg}(z)$ is defined to be the angle $\theta_0 \in (-\pi, \pi]$ representing the angle between the origin-z line and the real axis.



Definition 1.6. (argument map) The **argument map** is the map defined on $\mathbb{C} \setminus \{0\}$ by

$$\arg(z) = \{\mathrm{Arg}(z) + 2k\pi i : k \in \mathbb{Z}\}.$$

Proposition 1.7. (De Moivre's Theorem) For any $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$, we have

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Proof sketch: Induction and product to sum formula.

Definition 1.8. (roots of complex number) Given any $z \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$, the n-th **roots** of z is given by

$$z^{\frac{1}{n}} := \left\{ \sqrt[n]{|z|} \left(\cos\left(\frac{\mathrm{Arg}(z) + 2k\pi}{n} \right) + i\sin\left(\frac{\mathrm{Arg}(z) + 2k\pi}{n} \right) \right),\ k \in \{1, \cdots, n-1\} \right\}.$$

Remark 1.9. Notice that $z^{\frac{1}{n}}$ is **multivalued** and different from $\sqrt[n]{z}$.

2 Problems

1. **True or False**

   (a) For any $\theta \in \mathbb{R}$ and $q \in \mathbb{Q}$, we have $(\cos\theta + i\sin\theta)^q = \cos(q\theta) + i\sin(q\theta)$.

   (b) Suppose $a \in \mathbb{R}$, $\sqrt[n]{a} = a^{\frac{1}{n}}$.

2. (a) Let $z = \cos\theta + i\sin\theta$, where $\theta \in \mathbb{R}$. Find the four values of z such that $\mathrm{Im}(z^2 + \overline{z}) = 0$.

   (b) Let z_1, z_2 be two values of z obtained in (a) such that $\mathrm{Im}(z_1) < 0 < \mathrm{Im}(z_2)$. For any positive n, define $S_n = \sum_{r=1}^{n} \omega^r$, where $\omega = \frac{z_2}{z_1}$.

   (i) Prove that $\omega^3 = 1$.

   (ii) If n is a multiple of 3, prove that $S_n = 0$.

   (iii) Does there exist an integer m such that $(S_{2009} + S_{2010} + S_{2011})^m = 2$? Explain.

   (iv) Find all positive integers k such that $(S_n)^k + (S_{n+1})^k + (S_{n+2})^k = 2$ for any positive integer n.

3. Prove that
$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(n+1/2)\theta]}{2\sin(\theta/2)}.$$

4. Suppose P is a polynomial with real coefficients. Show that $P(z_0) = 0$ **iff** $P(\overline{z_0}) = 0$.

5. (a) Show that the n-th roots of 1 (other than 1 itself) satisfy the **cyclotomic equation**

$$z^{n-1} + z^{n-2} + \cdots + z + 1 = 0.$$

   (b) Suppose we consider the $n-1$ diagonals of a regular n-gon inscribed in a unit circle obtained by connecting one vertex with all the others. Show that the product of their lengths is n.