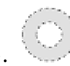
MATH4023
Tutorial 3 - Basic Point-set Topology, Complex-valued Function
March 10/12, 2020 Notes by Ivan So

1 Review

LANG(MATH)4023 vocabulary list:

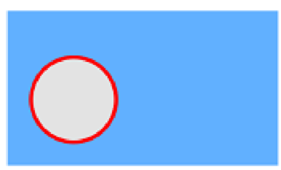
Special Domains	ball	annulus		
Nature of Points to a Set	interior	boundary	exterior	closure
Nature of Subset	connected	polygonally path-connected	simply-connected	convex
	open	closed	*compact*	*bounded*

Definition 1.1. (ball) A **ball** of *radius* R is the set $B_R(z_0) := \{z \in \mathbb{C} : |z - z_0| < R\}$. 

Definition 1.2. (annulus) An **annulus** is the set $A_{R,r}(z_0) := \{z \in \mathbb{C} : r < |z - z_0| < R\}$ for $0 \le r < R \le \infty$. 

Definition 1.3. (interior, boundary and exterior point) Consider $\Omega \subset \mathbb{C}$.

- $z \in \mathbb{C}$ is an **interior point** of Ω if there exists $\epsilon > 0$ such that $B_\epsilon(z) \subset \Omega$.
  Notation: Ω° = set of all interior points of Ω.
- $z \in \mathbb{C}$ is an **boundary point** of Ω if for any $\epsilon > 0$, $B_\epsilon(z) \cap \Omega \neq \emptyset$ and $B_\epsilon(z) \cap (\mathbb{C} \setminus \Omega) \neq \emptyset$.
  Notation: $\partial\Omega$ = set of all boundary points of Ω.
- $z \in \mathbb{C}$ is an **exterior point** of Ω if there exists $\epsilon > 0$ such that $B_\epsilon(z) \subset (\mathbb{C} \setminus \Omega)$.



The *gray* region denote the **interior**, the *red* for the **boundary** and the *blue* denote the **exterior**.

Definition 1.4. (closure) The **closure** $\overline{\Omega}$ of Ω is the union $\Omega \cup \partial\Omega$.

Definition 1.5. (open and closed subset) A subset Ω is said to be **open** if $\Omega = \Omega^\circ$. A subset C is said to be **closed** if $\partial C \subset C$ (or $\overline{\Omega} = \Omega$).

Proposition 1.6. For any $E \subset \mathbb{C}$, $\partial E \subset E$ **iff** $\mathbb{C} \setminus E$ is *open*.

Proposition 1.7. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence such that $z_n \in C$, a closed set. Suppose $\lim_{n\to\infty} z_n = w$, then $w \in E$.

Proposition 1.8. Union (finite or infinite) of open subset and finite intersection of open set is open. Finite union of closed subset and intersection (finite or infinite) of closed subset is closed.

Definition 1.9. (connected) A set Ω is said to be **connected** if for any pair of **open subsets** satisfying $\Omega \subset U \cup V$ and $U \cap V = \emptyset$, $\Omega \subset U$ or $\Omega \subset V$.

Definition 1.10. (polygonally path-connected) A set Ω is said to be **polygonally path connected** if any pair of points in Ω can be joined by a path consists of *finitely many* line segments.

Definition 1.11. (simply-connected) A set Ω is said to be **simply-connected** if it is

(i) *connected*;

(ii) every loop can contract continuously to a point without leaving Ω (no hole).

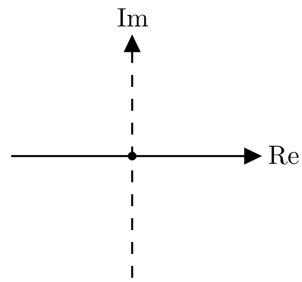
Definition 1.12. (convex domain) A subset $\Omega \subset \mathbb{C}$ is said to be **convex** if for any $z_1, z_2 \in \Omega$, there is a *straight line* connecting z_1 and z_2.

Definition 1.13. (complex-valued function) A **complex-valued function** is a map $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ for an *open* subset of $\mathbb{C}$.

2 Problems

1. **True or False**

   (a) The following is a simply-connected domain.



   True. A closed curve lying only in either $\{z : \mathrm{Re}\,(z) > 0\}$ or $\{z : \mathrm{Re}\,(z) < 0\}$ region surely can be contracted to a point. If it is a closed curve lying in both left $\{z : \mathrm{Re}\,(z) > 0\}$ and $\{z : \mathrm{Re}\,(z) < 0\}$ one can imagine the curve must be able to contract to a point since $z = 0$ is the only "channel". $\square$

   (b) There exists point $z \in \mathbb{C}$ that can be both an *interior point* and a *boundary point.*
   False. Suppose $z_0 \in \Omega^\circ$, then there exists ϵ_0 such that $B_{\epsilon_0}(z) \subset \Omega$. For such ϵ_0, $B_{\epsilon_0}(z) \cap \mathbb{C} \setminus \Omega = \emptyset$ by set definition. $\square$

2. Let

$$f(z) = \lim_{n\to\infty} \frac{z^n}{1+z^n}.$$

   (a) What is the domain of definition of f?

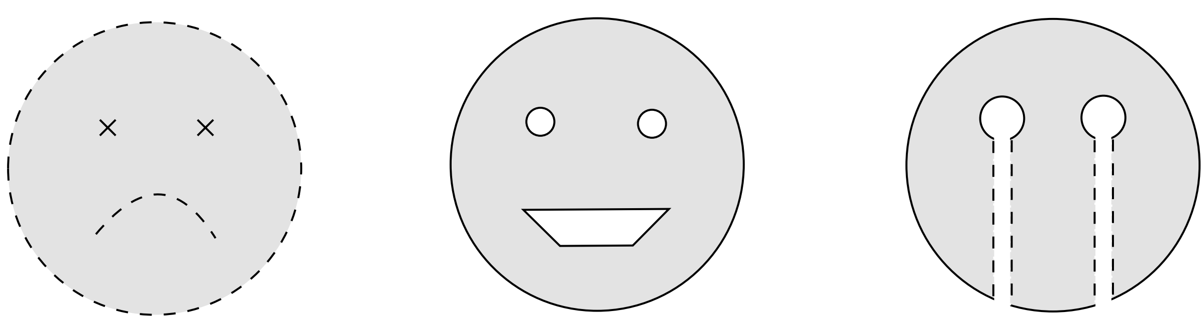
   (b) Give the explicit value for $f(z)$ for z in the defined domain.

Solution:

$$f(z) = \begin{cases} 0 & \text{if } |z| < 1 \text{ because } \lim_{n\to\infty} z^n = 0, \\ 1 & \text{if } |z| > 1 \text{ because } \lim_{n\to\infty} \frac{z^n}{z^n+1} = 1, \\ 1/2 & \text{if } |z| = 1 \text{ and } \mathrm{Arg}(z) = 0, \text{ obvious,} \\ \text{not defined,} & \text{if } |z| = 1 \text{ and } \mathrm{Arg}(z) \neq 0 \text{ because } \frac{\exp(in\mathrm{Arg}(z))}{1+\exp(in\mathrm{Arg}(z))} \text{ diverges as } n \to \infty. \end{cases}$$

$\square$

3. Below are your possible faces after MATH4023 midterm:



(a) immediately after exam　(b) result distributed, satisfied　(c) result distributed, really bad!!

Determine one by one whether they satisfy each of the following topological properties: (i) open (ii) closed (iii) connected (iv) polygonally path connected (v) simply-connected (vi) convex.

Solution:

	(a)	(b)	(c)
open	•		
closed		•	
connected	•	•	•
polygonally path-connected	•	•	•
simply connected			•
convex			

□

4. Let $f(z) = 1/z$. Describe what f does to points in inside, outside and on the unit circle ∂D.

Solution: The function is defined on $\mathbb{C} \setminus \{0\}$. Writing $z = re^{i\theta}$, we have

$$f(z) \in \begin{cases} \mathbb{C} \setminus \overline{D} & \text{if } z \in D \setminus \{0\} \\ \partial D & \text{if } z \in \partial D \\ D & \text{if } z \in \mathbb{C} \setminus \overline{D} \end{cases}$$

Meanwhile, the principle argument is flipped in sign by the inversion. □

5. Prove that every convex region is simply-connected.

Solution: Suppose $\Omega \subset \mathbb{C}$ is convex. Picking arbitrary point $z_0 \in \Omega$. For any other point $z \in \Omega$, the line segment $L(z_0, z)$ joining z_0 and z is contained in Ω (convexity). Now for any closed curve $\gamma \subset \Omega$, we can construct a contraction

$$\begin{aligned} C_\gamma : \gamma \times [0,1] &\to \Omega \\ (z,t) &\mapsto z + t(z - z_0) \end{aligned}$$

Convexity guarantee that while γ is contracting to a point, it always stay in Ω. Thereby proving the simply-connectivity. □