MATH4023
Tutorial 2 - Sequences and Series of Complex Numbers, Complex Exponential
March 3/5, 2020 Notes by Ivan So

1 Review

1.1 Complex Sequences

Definition 1.1. (limit of complex-valued sequence) Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. z_n *converges* to the **limit** L if for any $\epsilon > 0$, there exists N_ϵ such that

$$n > N_\epsilon \Longrightarrow |z_n - L| < \epsilon.$$

Proposition 1.2. A *complex sequence* converge to L **iff** $\operatorname{Re}(z)$ converges to $\operatorname{Re}(L)$ and $\operatorname{Im}(z)$ converges to $\operatorname{Im}(L)$.

Definition 1.3. A *complex sequence* is said to be **Cauchy** if for any $\epsilon > 0$, there exists N_ϵ such that

$$n, m > N_\epsilon \Longrightarrow |z_n - z_m| < \epsilon.$$

Theorem 1.4. (completeness of $\mathbb{C}$) For any *Cauchy* sequence over $\mathbb{C}$, there exists $L \in \mathbb{C}$ such that L is the limit of the sequence.

Proof sketch: Completeness of $\mathbb{R}$ + considering real and imaginary components individually.

1.2 Complex Series

Definition 1.5. (conditional and absolute convergence) Given a complex series $S = \sum_{n=1}^{\infty} z_n$. S is said to be **converge absolutely** if $\sum_{n=1}^{\infty} |z_n|$ converges. S is said to be **converge conditionally** if S converges but *does not converge absolutely.*

Proposition 1.6. Ratio test (for modulus), root test (for modulus) and absolute convergence test works for complex series.

Definition 1.7. The **radius of convergence** $R \in \mathbb{R}$ of a series $s(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is a value such that whenever $z \in B(z_0, R)$, $s(z)$ converges.

1.3 Exponential

Definition 1.8. The **complex exponential function** defined for any $z \in \mathbb{C}$ is

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Proposition 1.9. (Properties regarding complex exponential)

(a) (Euler's identity) $e^{i\theta} = \cos\theta + i \sin\theta$,

(a) $e^{z+w} = e^z e^w$.

2 Problems

1. **True or False**

   (a) We can conclude that the series defined by e^z has an infinite radius of convergence from both ratio test and root test.

   (b) Let T_n be the n-th term of the complex series. If $\lim_{n\to\infty} T_n = 0$, then $\sum_{n=1}^{\infty}(-1)^n T_n$ converges.

2. Let $a, b \in \mathbb{C}$. Assume that b not equal to any integer $0 \leq$. Show that the radius of convergence of the series

$$\sum \frac{a(a+1)\cdots(a+n)}{b(b+1)\cdots(b+n)} z^n$$

   is at least 1. Show that this radius can be infinity in some cases.

3. (a) Show that

$$S(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \begin{cases} \frac{1}{(1-z)^2} & \text{for } |z| < 1, \\ \frac{1}{z(1-z)^2} & \text{for } |z| > 1 \end{cases}.$$

(b) Prove that the convergence is uniform for $|z| \le c < 1$ in the first case, $|z| \ge b > 1$ in the second.

4. (Ratio test implies root test) Show that if $\{a_n\}_{n=0}^{\infty}$ is a sequence of non-zero complex numbers such that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$, then $\lim_{n\to\infty} |a_n|^{1/n} = L$.

5. Suppose $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ are two finite sequences of complex numbers . Let $B_k = \sum_{n=1}^k b_n$ denote the partial sums of the series $\sum b_n$ with the convention $B_0 = 0$. Prove the **summation by part formula**

$$\sum_{n=M}^{N} a_n b_n = a_N b_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n.$$