Claim: $|z+w| \le |z| + |w|$

Proof:

$$|z+w|^2 \qquad \left[\ \cdots \le (|z|+|w|)^2\ \right]$$

$$= (z+w)\overline{(z+w)}$$

$$= (z+w)(\bar{z}+\bar{w})$$

$$= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w}$$

$$= |z|^2 + z\bar{w} + \overline{z\bar{w}} + |w|^2$$

$$= |z|^2 + 2\,\mathrm{Re}(z\bar{w}) + |w|^2$$

$$\le |z|^2 + 2|z\bar{w}| + |w|^2$$

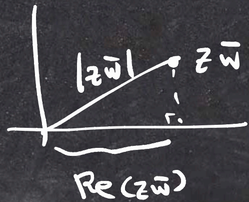
$$= |z|^2 + 2|z|\,|w| + |w|^2$$

$$= (|z|+|w|)^2$$

$$|z|^2 = z\bar{z}$$

$$z + \bar{z} = 2x = 2\,\mathrm{Re}(z).$$

$z = x+yi$, $\bar{z} = x-yi$



Cor of Δ-inequality:

$$\big||z|-|w|\big| \le |z-w|$$

$$z = w + (z-w)$$

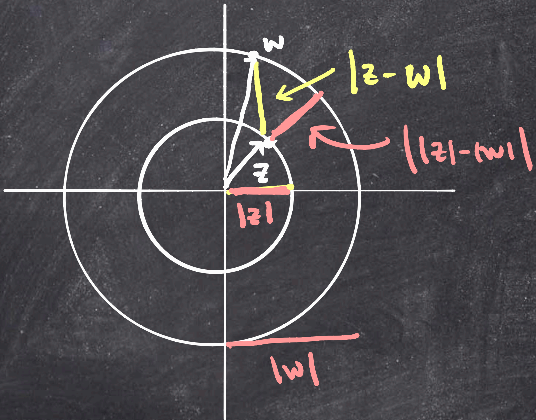
$$|z| \le |w| + |z-w|$$

$$|z|-|w| \le |z-w|.$$

$$\left|\frac{x+y}{x-y}\right| \le \frac{|x|+|y|}{\big||x|-|y|\big|}$$

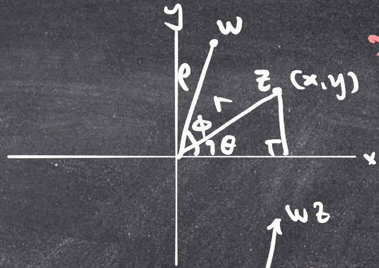
$$|x-y| \ge \big||x|-|y|\big| \ge |y|-|x|$$

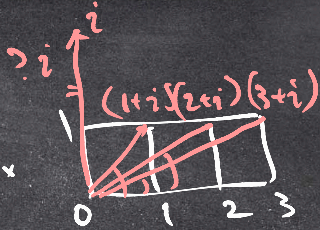
$$\ge |x|-|y|$$



$$z = x + yi$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$





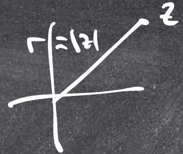
$$z = r\cos\theta + ir\sin\theta$$
$$= r\underbrace{(\cos\theta + i\sin\theta)}_{\text{cis}\,\theta}$$

$$w = \rho(\cos\phi + i\sin\phi)$$

$$zw = \rho r(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$$
$$= \rho r\begin{pmatrix}(\cos\theta\cos\phi - \sin\theta\sin\phi)\\ + i(\sin\theta\cos\phi + \cos\theta\sin\phi)\end{pmatrix} = \rho r\begin{pmatrix}\cos(\theta+\phi)\\ + i\sin(\theta+\phi)\end{pmatrix}$$

$$z = r(\cos\theta + i\sin\theta)$$

$|z|$



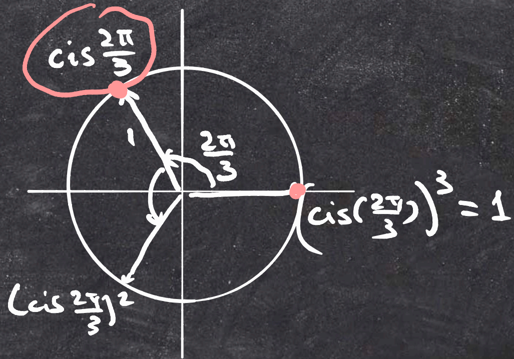
$$\operatorname{Arg} z = \theta \in (-\pi, \pi] \qquad . \qquad \arg z = \{\theta + 2n\pi : n \in \mathbb{Z}\}.$$

real: $x^3 = 1 \Rightarrow x = 1$.

complex: $z^3 = 1$

$$\left(\operatorname{cis}\frac{2\pi}{3}\right)^3 = \operatorname{cis}\left(\frac{2\pi}{3}\cdot 3\right) = \operatorname{cis} 2\pi$$
$$= 1$$

$$\left(\operatorname{cis}\frac{4\pi}{3}\right)^3 = \operatorname{cis} 4\pi = 1.$$



~~$1^{\frac{1}{3}} = 1$ (Real world)~~

$$1^{\frac{1}{3}} = \left\{ 1,\ \operatorname{cis}\frac{2\pi}{3},\ \operatorname{cis}\frac{4\pi}{3} \right\}.$$

~~$1^{\frac{1}{2}} = 1$ (real)~~

$1^{\frac{1}{2}}$ = all roots $\boxed{z^2 = 1}$ $= \{1, -1\}$.

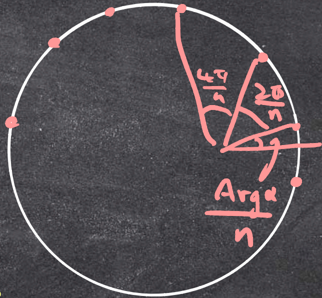
$$1^{\frac{1}{n}} = \left\{ \operatorname{cis}\left(\frac{2\pi}{n}\cdot k\right) : k = 0, 1, \dots, n-1 \right\}$$

$\alpha^{\frac{1}{n}}$ = roots of $\left(\underline{z^n = \alpha}\right)$

$$(r \operatorname{cis}\theta)^n = \alpha = |\alpha| \operatorname{cis}\phi.$$

? ?

$$r = \sqrt[n]{|\alpha|}, \qquad \theta = \frac{\phi}{n}$$



Check: $\left(\sqrt[n]{|\alpha|} \operatorname{cis}\frac{\phi}{n}\right)^n = |\alpha| \operatorname{cis}\phi = \alpha.$

Also: $\left(\sqrt[n]{|\alpha|} \operatorname{cis}\frac{\phi + 2k\pi}{n}\right)^n = |\alpha| \operatorname{cis}(\phi + 2k\pi)$

$$= |\alpha| \operatorname{cis}\phi = \alpha.$$

$$\alpha^{\frac{1}{n}} = \left\{ \sqrt[n]{|\alpha|} \operatorname{cis}\frac{\operatorname{Arg}\alpha + 2k\pi}{n} : k = 0, 1, \ldots, n-1 \right\}.$$