Part A

1. C

2. A, B, D

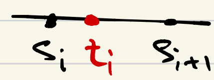
3. B, C (A accepted only if your own HW4 used Vitali's 3-cover to prove Vitali 5-cover)

4. let $S=\{s_n\}_{n=1}^{\infty}$, where $n \mapsto s_n$ is injective.

For each $N\in\mathbb{N}$, we consider $\{s_1,\dots,s_N\}$.
By reordering, assume $s_1<s_2<\dots<s_N$.

$\forall i=1,2,\dots,N-1$ $\exists\, t_i\in(s_i,s_{i+1})\cap S^c$ (otherwise $(s_i,s_{i+1})\subset S$ then S is uncountable.)

Consider the mesh:



$$\{s_1<t_1<s_2<t_2<s_3<\dots<s_{N-1}<t_{N-1}<s_N\}$$

$$|\chi_S(s_1)-\chi_S(t_1)|+|\chi_S(t_1)-\chi_S(s_2)|+\dots+|\chi_S(t_{N-1})-\chi_S(s_N)|$$

$$=|1-0|+|0-1|+\dots+|0-1|$$

$$=2(N-1)$$

$$\Rightarrow T_{\chi_S}(0,1)=\sup_{\substack{P \text{ partition}\\ \text{of } [0,1]}}\sum_i|\chi_S(x_i)-\chi_S(x_{i-1})|\ge 2(N-1)\quad \forall N\in\mathbb{N}.$$

$$\Rightarrow T_{\chi_S}(0,1)=+\infty. \quad \blacksquare$$

5. Note $-\frac{x^2-y^2}{x^2+y^2}$ is continuous on $(0,1]\times(0,1]$

$$\Rightarrow \frac{\partial^2}{\partial x\partial y}\tan^{-1}\frac{y}{x}=\frac{\partial^2}{\partial y\partial x}\tan^{-1}\frac{y}{x}$$

$$\int_{[0,1]}\left(\int_{[0,1]}f(x,y)\,dx\right)dy=\int_{[0,1]}\left(\int_{[0,1]}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\tan^{-1}\frac{y}{x}\right)dx\right)dy$$

$$=\int_{[0,1]}\left[\frac{\partial}{\partial y}\tan^{-1}\frac{y}{x}\right]_{x=0}^{x=1}dy=\int_{[0,1]}\left[\frac{1}{1+(\frac{y}{x})^2}\cdot\frac{1}{x}\right]_{x=0}^{x=1}dy=\int_{[0,1]}\left[\frac{x}{x^2+y^2}\right]_{x=0}^{x=1}dy$$

$$=\int_{[0,1]}\frac{1}{1+y^2}\,dy=\left[\tan^{-1}y\right]_{y=0}^{y=1}=\frac{\pi}{4}.$$

However,

$$\int_{[0,1]}\left(\int_{[0,1]} f(x,y)\,dy\right)dx = \int_{[0,1]}\left(\int_{[0,1]} \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\tan^{-1}\frac{y}{x}\right)dy\right)dx$$

$$= \int_{[0,1]} \left[\frac{\partial}{\partial x}\tan^{-1}\frac{y}{x}\right]_{y=0}^{y=1} dx$$

$$= \int_{[0,1]} \left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right)\right]_{y=0}^{y=1} dx$$

$$= \int_{[0,1]} -\frac{1}{1+x^2}\,dx = \left[-\tan^{-1}x\right]_{x=0}^{x=1} = -\frac{\pi}{4}.$$

$$\therefore \int_{[0,1]}\left(\int_{[0,1]} f(x,y)\,dx\right)dy \neq \int_{[0,1]}\left(\int_{[0,1]} f(x,y)\,dy\right)dx$$

$$\Rightarrow \int_{[0,1]\times[0,1]} |f(x,y)|\,dxdy = \infty$$ (contrapositive of Fubini's Theorem)

Part B

1(a) For any (a,b), $\exists\, r_n \in \mathbb{Q}\cap(a,b)$. Note that

$\lim_{x\to r_n^+} f(x-r_n) = \lim_{x\to r_n^+} \frac{1}{\sqrt{x-r_n}} = +\infty$, so given any $N\in\mathbb{N}$,

$\exists\, x_N \in (r_n, b)$ s.t. $f(x_N - r_n) \geq N \Rightarrow g(x_N) \geq \frac{1}{2^n} f(x_N - r_n)$

$$\geq \frac{N}{2^n}$$

$$\therefore \sup_{x\in(a,b)} g(x) \geq g(x_N) \geq \frac{N}{2^n} \quad \forall N\in\mathbb{N}$$

$$\Rightarrow \sup_{x\in(a,b)} g(x) = +\infty.$$

(b) Note $f\geq 0$, by Monotone Convergence Theorem

$$\int_{\mathbb{R}} g(x)\,dx = \int_{\mathbb{R}} \sum_{n=1}^{\infty} \frac{1}{2^n} f(x-r_n) = \sum_{n=1}^{\infty}\frac{1}{2^n}\int_{\mathbb{R}} f(x-r_n)\,dx.$$

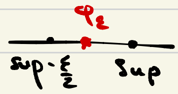
By translation invariance, $\int_{\mathbb{R}} f(x-r_n)\,dx = \int_{\mathbb{R}} f(x)\,dx = \int_{(0,1)} \frac{1}{\sqrt{x}}\,dx = [2\sqrt{x}]_0^1 = 2$

$$\therefore \int_{\mathbb{R}} g(x)\,dx = \sum_{n=1}^{\infty}\frac{1}{2^n}\cdot 2 = \sum_{n=1}^{\infty}\frac{1}{2^{n-1}} = \frac{1}{1-\frac{1}{2}} = 2 \quad \blacksquare$$

2(a) $$\int_{[0,1]} h \, d\mu = \sup_{\substack{0 \le \varphi \le h \\ \varphi \text{ bounded}}} \int_{[0,1]} \varphi \, d\mu .$$

$\therefore \forall \varepsilon > 0$, $\exists \varphi_\varepsilon : [0,1] \to [0,\infty)$, bounded and $0 \le \varphi_\varepsilon \le h$
s.t.
$$\int_{[0,1]} h \, d\mu - \frac{\varepsilon}{2} < \int_{[0,1]} \varphi_\varepsilon \, d\mu \le \int_{[0,1]} h \, d\mu .$$

$$\Rightarrow \int_{[0,1]} |h - \varphi_\varepsilon| \, d\mu = \int_{[0,1]} (h - \varphi_\varepsilon) \, d\mu < \frac{\varepsilon}{2}$$



$h \ge \varphi_\varepsilon$.

$\forall A \subset [0,1]$ measurable :

$$\therefore \int_A h \, d\mu = \int_A |h| \, d\mu \le \int_A |h - \varphi_\varepsilon| + \varphi_\varepsilon \, d\mu$$
$$\le \int_{[0,1]} |h - \varphi_\varepsilon| \, d\mu + \int_A \sup_{[0,1]} \varphi_\varepsilon \, d\mu$$
$$< \frac{\varepsilon}{2} + \sup_{[0,1]} \varphi_\varepsilon \cdot \mu(A)$$

$\therefore$ Whenever $\mu(A) < \boxed{\delta := \dfrac{\varepsilon}{2\left(\sup_{[0,1]} \varphi_\varepsilon\right) + 1}}$ we have

$$\int_A h \, d\mu < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

■

(b) $f_n \in AC[0,1] \Rightarrow f_n(x+h) - f_n(x) = \int_{[x,x+h]} f_n' \, d\mu \quad \forall n \in \mathbb{N}$, — (1)

letting $n \to \infty$, $\lim_{n \to \infty} \left(f_n(x+h) - f_n(x) \right) = f(x+h) - f(x)$. — (2)

Also:

$$\left| \int_{[x,x+h]} f_n' \, d\mu - \int_{[x,x+h]} g \, d\mu \right| \le \int_{[x,x+h]} |f_n' - g| \, d\mu \le \| f_n' - g \|_{L^1} \to 0 .$$

$$\Rightarrow \lim_{n \to \infty} \int_{[x,x+h]} f_n' \, d\mu = \int_{[x,x+h]} g \, d\mu . \quad \text{— (3)}$$

Combining (1), (2), (3), we have:

$$f(x+h) - f(x) = \int_{[x,x+h]} g \, d\mu \qquad \forall x, \ x+h \in [0,1].$$

$$\Rightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h} \int_{[x,x+h]} g \, d\mu = g(x) \quad \underline{\forall \text{ a.e. } x \in [0,1]}$$

Lebesgue's Differentiation Theorem.

■

(c) Verify that $\|\cdot\|$ is a norm:

(i) Clearly $\|f\| \geq 0$. If $\|f\| = 0$, then $\sup_{x\in[0,1]} |f(x)| = 0$

$$\Rightarrow f(x) = 0 \quad \forall x \in [0,1]$$
$$\Rightarrow f = 0.$$

(ii) $\forall c \in \mathbb{R}$, $\|cf\| = \sup_{x\in[0,1]} |cf(x)| + \sup_{\substack{\text{partitions}\\ \text{of } [0,1]}} \sum_{i=1}^{N} |cf(x_i) - cf(x_{i-1})|$

$$= \sup_{x\in[0,1]} |c||f(x)| + \sup_{\substack{\text{partitions}\\ \text{of } [0,1]}} \sum_{i=1}^{N} |c||f(x_i) - f(x_{i-1})|$$

$$= |c| \left(\sup_{x\in[0,1]} |f(x)| + \sup_{\substack{\text{partitions}\\ \text{of } [0,1]}} \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| \right)$$

$$= |c| \|f\|.$$

(iii) $\|f+g\| = \sup_{x\in[0,1]} |f(x) + g(x)| + \sup_{\substack{\text{partitions}\\ \text{of } [0,1]}} |(f+g)(x_i) - (f+g)(x_{i-1})|$

$$\leq \sup_{x\in[0,1]} (|f(x)| + |g(x)|) + \sup_{\substack{\text{partitions}\\ \text{of } [0,1]}} (|f(x_i) - f(x_{i-1})| + |g(x_i) - g(x_{i-1})|)$$

$$\leq \underbrace{\sup_{x\in[0,1]} |f(x)| + \sup_{\substack{\text{partitions}\\ \text{of } [0,1]}} |f(x_i) - f(x_{i-1})|}_{= \|f\|} + \underbrace{\sup_{x\in[0,1]} |g(x)| + \sup_{\substack{\text{partitions}\\ \text{of } [0,1]}} |g(x_i) - g(x_{i-1})|}_{= \|g\|}$$

$$= \|f\| + \|g\|.$$

$\therefore$ $\|\ \|$ is a norm on $AC[0,1]$.

Next we show $\|\cdot\|$ is a complete norm.

let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence wrt $\|\cdot\|$.

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $m, n > N \Rightarrow \|f_m - f_n\| < \varepsilon$

$$\Rightarrow \begin{cases} \sup_{x\in[0,1]} |f_m(x) - f_n(x)| \leq \|f_m - f_n\| < \varepsilon & \text{— }(*) \\ T_{f_m - f_n}(0,1) \leq \|f_m - f_n\| < \varepsilon. \end{cases}$$

$(*) \Rightarrow \{f_n\}$ is a Cauchy sequence wrt sup-norm

$\Rightarrow \exists f : [0,1] \to \mathbb{R}$ s.t. $f_n \rightrightarrows f$ on $[0,1]$.

We then need to show $\underline{f \in AC[0,1]}$ and $\underline{f_n \to f \text{ in } \|\cdot\|\text{-norm}}$.

From HW4, $f_m - f_n \in AC[0,1]$

$$\Rightarrow T_{f_m - f_n}(0,1) = \int_0^1 |f_m' - f_n'| \, d\mu = \|f_m' - f_n'\|_{L^1}.$$

$\therefore$ (*) $\Rightarrow \{f_n'\}$ is a Cauchy sequence wrt L^1-norm

L^1 is complete $\Rightarrow \exists g \in L^1[0,1]$ s.t. $f_n' \to g$ in L^1-norm —— (**)

$$\begin{cases} f_n \rightrightarrows f \text{ on } [0,1] \\ \|f_n' - g\|_{L^1[0,1]} \to 0 \end{cases} \xRightarrow{(b)} f' = g \text{ a.e. on } [0,1].$$

From (a), $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\mu(A) < \delta \Rightarrow \int_A |f'| \, d\mu < \varepsilon$.

$\therefore$ For any disjoint intervals $(x_1, x_2), \ldots, (x_N, x_{N+1})$

s.t. $\underbrace{\sum_{i=1}^N |x_i - x_{i+1}|}_{= \mu\left(\coprod_{i=1}^N (x_i, x_{i+1})\right)} < \delta$, we have $\int_{\coprod_{i=1}^N (x_i, x_{i+1})} |f'| \, d\mu < \varepsilon$

$$\Rightarrow \sum_{i=1}^N \int_{(x_i, x_{i+1})} |f'| \, d\mu < \varepsilon \quad \text{—— } (\dagger)$$

$$\therefore \sum_{i=1}^N |f(x_i) - f(x_{i+1})| = \lim_{n\to\infty} \sum_{i=1}^N |f_n(x_i) - f_n(x_{i+1})|$$

$$= \lim_{n\to\infty} \sum_{i=1}^N \left| \int_{(x_i, x_{i+1})} f_n' \, d\mu \right| \quad (\text{as } f_n \in AC[0,1])$$

$$\le \overline{\lim_{n\to\infty}} \sum_{i=1}^N \int_{(x_i, x_{i+1})} |f_n'| \, d\mu.$$

$$= \sum_{i=1}^N \int_{(x_i, x_{i+1})} |f'| \, d\mu \quad \text{as } f_n' \to f' \text{ in } L^1[0,1].$$

$$< \varepsilon \quad (\text{from } (\dagger))$$

$\therefore f \in AC[0,1]$.

Lastly to show $\|f_n - f\| \to 0$, we note that

$$\|f_n - f\| = \underbrace{\sup_{x \in [0,1]} |f_n(x) - f(x)|}_{\substack{\downarrow \\ 0}} + \underbrace{\|f_n' - f'\|_{L^1[0,1]}}_{\text{from } f_n, f \in AC[0,1] \text{ and HW4.}}$$

by uniform convergence

$= \|f_n' - g\|_{L^1[0,1]} \to 0$ from (**).

$\therefore \|f_n - f\| \to 0$.

Hence $\{f_n\} \to f$ in $\|\cdot\|$-norm $\forall$ Cauchy-seq. $\{f_n\}$.

$\therefore (AC[0,1], \|\cdot\|)$ is a Banach space.

3(a) Suppose $y \in \bigcup_{n=1}^{\infty} K_n$, we need to show $y \in \mathcal{U}$.

$\exists\, n_0 \in \mathbb{N}$ s.t. $y \in K_{n_0}$.

$\therefore\ |y| \leq n_0$ and $\underbrace{y \notin \bigcup_{x \in \mathbb{R}^d - \mathcal{U}} B(x, \tfrac{1}{n_0})}$

$\Rightarrow \forall x \in \mathbb{R}^d - \mathcal{U},\ y \notin B(x, \tfrac{1}{n_0})$

$\Rightarrow \forall x \in \mathbb{R}^d - \mathcal{U},\ |y - x| \geq \tfrac{1}{n_0}$.

$\therefore\ y \in \mathcal{U}$ (otherwise, take $x = y$, then $0 = |y - x| \geq \tfrac{1}{n_0}$ ~~$\cancel{\rightarrow\leftarrow}$~~).

Conversely, suppose $y \in \mathcal{U}$, need to show $y \in K_n\ \exists n \in \mathbb{N}$.

Suppose NOT: $y \notin K_n\ \forall n \in \mathbb{N}$.

then $\forall n \in \mathbb{N}$, $|y| > n$ or $y \in \bigcup_{x \in \mathbb{R}^d - \mathcal{U}} B(x, \tfrac{1}{n})$

$\Rightarrow \underbrace{|y| > n}_{\text{I}}$ or $\underbrace{\exists x_n \in \mathbb{R}^d - \mathcal{U} \text{ s.t. } |x_n - y| < \tfrac{1}{n}}_{\text{II}}$.

If II holds for infinitely many n's, then $\exists \{x_{n_k}\}_{k=1}^{\infty} \subset \mathbb{R}^d - \mathcal{U}$ s.t. $|x_{n_k} - y| < \tfrac{1}{n_k}$.

$\therefore\ y = \lim_{k \to \infty} x_{n_k}$. Also, $\mathbb{R}^d - \mathcal{U}$ is closed, so $\{x_{n_k}\} \subset \mathbb{R}^d - \mathcal{U}$

$\Rightarrow y \in \mathbb{R}^d - \mathcal{U}$

$\Rightarrow y \notin \mathcal{U}$

$\therefore$ a contradiction

$\therefore$ II only holds for finitely many n's.

$\Rightarrow$ I holds for infinitely many n's.
But it would show $|y| = \infty$, impossible.

$\therefore$ Assumption $y \notin K_n\ \forall n$ is impossible

$\Rightarrow y \in \bigcup_{n=1}^{\infty} K_n$.

$\therefore\ \mathcal{U} = \bigcup_{n=1}^{\infty} K_n$.

(b) Similar to the proof of Hardy-Littlewood's inequality:

E_α is open by definition.

let $K_n := \overline{B(0,n)} \cap \Big(\mathbb{R}^d - \underbrace{\bigcup_{x\in\mathbb{R}^d-U} B(x,\tfrac{1}{n})}_{\text{open}}\Big)$

then (a) $\Rightarrow$ $\bigcup_{n=1}^{\infty} K_n = E_\alpha$.

Note also that $K_1 \subset K_2 \subset K_3 \subset \cdots$ from definition.

Consider a fixed K_n (which is compact = closed and bounded):

$\forall x \in K_n \subset E_\alpha$, definition of E_α

$$\Rightarrow \exists B_x \ni x \text{ s.t. } \frac{\int_{B_x} |f| \varphi \, d\mu}{\int_{B_x} \varphi \, d\mu} > \alpha. \quad \text{——} \ (*)$$

$$\left\{\begin{array}{l} K_n \subset \bigcup_{x\in K_n} B_x \\ K_n \text{ compact} \end{array}\right. \xrightarrow{\text{Heine-Borel}} \exists x_1, \ldots, x_N \in K_n \text{ s.t. } K_n \subset B_{x_1} \cup \cdots \cup B_{x_N}.$$

Vitali's 3-Covering lemma

$\Rightarrow \exists\, x_{i_1}, \ldots, x_{i_m} \subset \{x_1, \ldots, x_N\}$ s.t.

$K_n \subset \bigcup_{i=1}^{N} B_{x_i} \subset \bigcup_{j=1}^{m} 3B_{x_{i_j}}$ and $\{B_{x_{i_j}}\}_{j=1}^{m}$ are pairwise disjoint.

$$\therefore \int_{K_n} \varphi \, d\mu \le \int_{\bigcup_{j=1}^{m} 3B_{x_{i_j}}} \varphi \, d\mu \le \sum_{j=1}^{m} \int_{3B_{x_{i_j}}} \varphi \, d\mu \quad \text{——} \ (\dagger)$$

Note that

$$\int_{\underset{=\gamma\cdot\frac{3}{\gamma}}{3B}} \varphi \, d\mu \le C(d,\gamma) \int_{\frac{3}{\gamma}B} \varphi \, d\mu \le C(d,\gamma)^2 \int_{\frac{3}{\gamma^2}B} \varphi \, d\mu$$

$$\le \cdots \le C(d,\gamma)^k \int_{\frac{3}{\gamma^k}B} \varphi \, d\mu.$$

Take k large s.t. $\frac{3}{\gamma^k} \le 1$, then
(k depends only on γ.)

$$\int_{3B} \varphi \, d\mu \le C(d,\gamma)^k \int_{B} \varphi \, d\mu.$$

as $B \supset \frac{3}{\gamma^k} B$.

Relabel $\tilde{C}(d,\gamma) = C(d,\gamma)^k$.

Back to (†): we have

$$\int_{K_n} \varphi \, d\mu \le \sum_{j=1}^{m} \tilde{C}(d,r) \int_{B_{x_j}} \varphi \, d\mu$$

$$\le \sum_{j=1}^{m} \tilde{C}(d,r) \frac{1}{\alpha} \int_{B_{x_j}} |f| \varphi \, d\mu \qquad (\text{from } (*))$$

$$= \frac{\tilde{C}(d,r)}{\alpha} \int_{\coprod_{j=1}^{m} B_{x_j}} |f| \varphi \, d\mu \le \frac{\tilde{C}(d,r)}{\alpha} \int_{\mathbb{R}^d} \underbrace{|f|}_{L^1(\mathbb{R}^d)} \underbrace{\varphi}_{\text{bounded}} \, d\mu \quad \text{— } (✡)$$

$$\Rightarrow |f|\varphi \in L^1(\mathbb{R}^d).$$

Finally, $\displaystyle\int_{K_n} \varphi \, d\mu = \int_{E_\alpha} \chi_{K_n} \varphi \, d\mu$ as $K_n \subset E_\alpha$.

$K_n \subset K_{n+1} \ \forall n \Rightarrow \chi_{K_n} \nearrow$ and $\displaystyle\lim_{n\to\infty} \chi_{K_n} = \chi_{\bigcup_{n=1}^{\infty} K_n} = \chi_{E_\alpha}$.

$$\therefore \lim_{n\to\infty} \int_{K_n} \varphi \, d\mu \underset{\uparrow \text{ MCT}}{=} \int_{E_\alpha} \lim_{n\to\infty} \chi_{K_n} \varphi \, d\mu = \int_{E_\alpha} \chi_{E_\alpha} \varphi \, d\mu = \int_{E_\alpha} \varphi \, d\mu.$$

Combining with (✡), we have

$$\int_{E_\alpha} \varphi \, d\mu = \lim_{n\to\infty} \int_{K_n} \varphi \, d\mu \le \frac{\tilde{C}(d,r)}{\alpha} \int_{\mathbb{R}^d} |f| \varphi \, d\mu$$ as desired. ∎