香港科技大學
THE HONG KONG
UNIVERSITY OF SCIENCE
AND TECHNOLOGY

數學系
DEPARTMENT OF MATHEMATICS

FINAL EXAMINATION

Course Code: MATH 3043
Course Title: Honors Real Analysis
Semester: Fall 2019-20
Date and Time: 4:30PM-7:30PM, 14 December 2019

Instructions

- It is an **OPEN-NOTES** exam. You can look at any materials, both online and offline. However, only results discussed in lectures, or proved in homework can be directly quoted.
- Discussion with any person (online or offline) is **strictly prohibited**, and is a serious violation of the honor code. Posting related questions in any online forum, whether it is answered or not, is also a serious violation of the honor code.
- Answer **ALL** problems. Write your solutions in your own paper. Submit the file as a PDF in Canvas, or turn it in personally to Room 3488 by 7:45PM today.
- You must **SHOW YOUR WORK**, **JUSTIFY YOUR ARGUMENTS**, and **PRESENT CLEARLY** in order to receive credits in every problem in Part B.
- Some problems in Part B are structured into several parts. You can quote the results stated in the preceding parts to do the next part.

HKUST Academic Honor Code

Honesty and integrity are central to the academic work of HKUST. Students of the University must observe and uphold the highest standards of academic integrity and honesty in all the work they do throughout their program of study. As members of the University community, students have the responsibility to help maintain the academic reputation of HKUST in its academic endeavors. Sanctions will be imposed on students, if they are found to have violated the regulations governing academic integrity and honesty.

"I confirm that I have answered the questions using only materials specified approved for use in this examination, that all the answers are my own work, and that I have not received any assistance during the examination."

Student's Signature: ________________________________

Student's Name: ________________________________
FAMILY NAME, First Name

HKUST ID: ____________________ **Seat Number:** ____________

Part A - Short Questions (20 points)

[Recommended time: < 30m]
Write your solution in your own paper. **Clearly state the question number of each question**. For multiple choice, just write down the letters A, B, C, etc.

1. Let $f : \mathbb{R}^d \to \mathbb{R}$ be in $L^1(\mathbb{R}^d)$. What is the following function called? **[1]**

$$Mf(x) = \sup_{x \in B} \frac{1}{\mu(B)} \int_B f \, d\mu.$$

   Write down the correct answer:

   A. Holy-Littlewood's maximal function
   B. Little-Hollywood's maximal function
   C. Hardy-Littlewood's maximal function
   D. Little-Hyperwood's maximal function
   E. Happy-Neighborhood's maximal function
   F. Highly-Likelihood's maximal function

2. Which condition(s) below guarantee that the function f is differentiable a.e. in $\mathbb{R}$? Write down ALL correct answer(s): **[4]**

   A. f is of bounded variations on any closed and bounded interval in $\mathbb{R}$.
   B. f is absolutely continuous on any closed interval of length 1 in $\mathbb{R}$.
   C. f is locally L^1 in $\mathbb{R}$.
   D. f equals the Cantor-Lebesgue's function (devil's staircase) on $[0, 1)$, and is a periodic function with period 1.

3. Based on the proofs done in **lectures or homework**, which of the following is/are consequence(s) of Vitali's 3-Covering Lemma? Write down ALL correct answer(s): **[4]**

   A. Vitali's 5-Covering Lemma
   B. Lebesgue's Differentiation Theorem
   C. $\displaystyle\int_a^b f'(x) \, dx = f(b) - f(a)$ for any $f \in AC[a, b]$
   D. Fubini-Tonelli's Theorem

   [Note: If (A) is proved using (B), and (B) is a consequence of Vitali's 3-Covering Lemma, then (A) is also considered as a consequence of Vitali's 3-Covering Lemma.]

4. Suppose S is a countable and infinite set in $[0, 1]$. Show that χ_S is NOT of bounded variations on $[0, 1]$. **[5]**

5. Consider the function $f : (0, 1] \times (0, 1] \to \mathbb{R}$ defined as: **[6]**

$$f(x, y) := \frac{\partial^2}{\partial x \partial y} \tan^{-1} \frac{y}{x} = -\frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

   Using the Fubini's Theorem, show that f is NOT Lebesgue integrable on $(0, 1] \times (0, 1]$.

Part B - Long Questions (80 points): Answer ALL problems

[Recommended time: Q1 $<$ 30m; Q2 $<$ 1h; Q3 $<$ 1h]
Assumption: Unless otherwise is stated, the measure μ means Lebesgue measure, and measurable sets and functions are with respect to the Lebesgue measure.

1. Let $f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$. Express $\mathbb{Q} = \{r_n\}_{n=1}^{\infty}$, where $n \mapsto r_n$ is injective. Consider the function $g : \mathbb{R} \to [0, \infty)$
$$g(x) := \sum_{n=1}^{\infty} \frac{1}{2^n} f(x - r_n).$$
   (a) Show that for any non-empty open interval (a,b), we have $\sup_{x \in (a,b)} g(x) = +\infty$. **[7]**

   (b) Find the exact value of $\int_{\mathbb{R}} g \, d\mu$. Justify your answers. **[8]**

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2. (a) Let $h : [0,1] \to [0, \infty)$ be a Lebesgue integrable function (h may be unbounded). Prove that for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $A \subset [0,1]$ is a measurable set with $\mu(A) < \delta$, we have $\int_A h \, d\mu < \varepsilon$. **[10]**
   [Hint: First consider the special case when h is bounded on $[0,1]$.]

   (b) Denote by $AC[0,1]$ the vector space of absolutely continuous functions on $[0,1]$. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence in $AC[0,1]$ such that f_n converges uniformly on $[0,1]$ to a function f, and f_n' converges in $L^1[0,1]$-norm to a function g. Show that $f' = g$ a.e. on $[0,1]$. **[10]**

   (c) For any $f \in AC[0,1]$, we define $\|f\| := \sup_{x \in [0,1]} |f(x)| + T_f(0,1)$. **[15]**

   Using the above results, prove that $AC[0,1]$ with the norm $\|\cdot\|$ is a Banach space.
   [No need to verify that $AC[0,1]$ is a vector space, but NEED to prove $\|\cdot\|$ is a norm.]

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3. (a) Let U be a non-empty open set in $\mathbb{R}^d$. Denote by $B(x,r)$ the open ball centered at x with radius r. Consider the sequence of sets: **[10]**
$$K_n := \overline{B(x,n)} \cap \left(\mathbb{R}^d - \bigcup_{x \in \mathbb{R}^d - U} B\big(x, \frac{1}{n}\big) \right).$$
   Show that $\bigcup_{n=1}^{\infty} K_n = U$.

   (b) Given that $\varphi(x) : \mathbb{R}^d \to (0, \infty)$ is a positive, measurable, bounded function on $\mathbb{R}^d$ satisfying the following condition: there exists a fixed constant $\gamma > 1$, and a constant $C(d, \gamma) > 1$ depending only on the dimension d and the constant γ, such that for any open ball $B \subset \mathbb{R}^d$, we have: **[20]**
$$\int_{\gamma B} \varphi \, d\mu \le C(d,\gamma) \int_B \varphi \, d\mu.$$
   Here γB means $B(x_0, \gamma r)$ if $B = B(x_0, r)$.

Given $f \in L^1(\mathbb{R}^d)$, and define for each $\alpha > 0$ the set

$$E_\alpha := \left\{ x \in \mathbb{R}^d : \sup_{B \ni x} \frac{\int_B |f| \, \varphi \, d\mu}{\int_B \varphi \, d\mu} > \alpha \right\}.$$

Prove that there exists a constant $\widetilde{C}(d, \gamma) > 1$ depending only on d and γ such that

$$\int_{E_\alpha} \varphi \, d\mu \leq \frac{\widetilde{C}(d, \gamma)}{\alpha} \int_{\mathbb{R}^d} |f| \, \varphi \, d\mu.$$

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* End of Paper *

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