Last time:

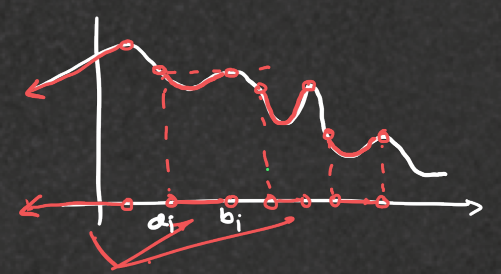
$F \in BV[a,b]$, then $F(x) = \underbrace{F(a) + P_F(a,x)}_{\text{monotone}\nearrow} - \underbrace{N_F(a,x)}_{\text{monotone}\nearrow}$

Goal: F is differentiable a.e. on $[a,b]$.

Extra assumption: F is continuous on $[a,b]$ (monotone)

Lemma 3.5

Given: $G:\mathbb{R}\to\mathbb{R}$ continuous.



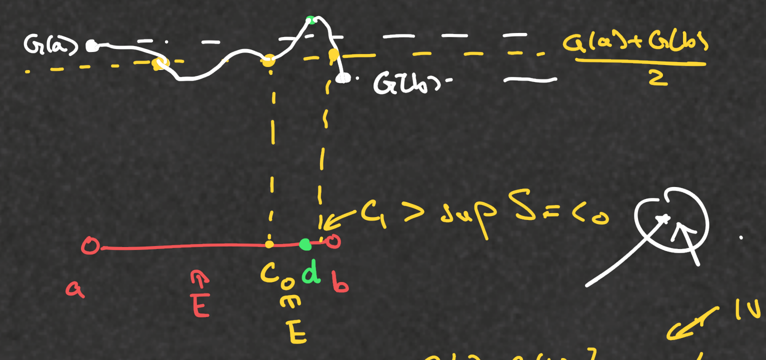
seaview.

$E := \{x\in\mathbb{R} : G(y) > G(x) \text{ for some } y > x\}$

Then: E is open

Write $E = \coprod_{i=1}^{\infty} (a_i, b_i)$, then, for those finite interval (a_i, b_i), we have $G(a_i) = G(b_i)$

Proof: Assume: $G(a) > G(b)$



$S = \{c \in (a,b) : G(c) = \frac{G(a)+G(b)}{2}\} \neq \emptyset$

Consider $\sup S = c_0$.

$\lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$ exists $\iff$ $D^+F(x) = D_+F(x) = D^-F(x) = D_-F(x) \neq \infty$.

$$D^+F(x) = \lim_{\delta\to 0^+} \sup_{h\in(0,\delta)} \frac{F(x+h)-F(x)}{h} \qquad \left(\overline{\lim_{h\to 0^+}} \frac{F(x+h)-F(x)}{h}\right)$$

$$D_+F(x) = \lim_{\delta\to 0^+} \inf_{h\in(0,\delta)} \frac{F(x+h)-F(x)}{h}$$

$$D^-F(x) = \lim_{\delta\to 0^+} \sup_{h\in(-\delta,0)} \frac{F(x+h)-F(x)}{h}$$

$$D_-F(x) = \lim_{\delta\to 0^+} \inf_{h\in(-\delta,0)} \frac{F(x+h)-F(x)}{h}$$

Claim: (i) $D^+F(x) < \infty$ for a.e. x

(ii) $D^+F(x) \le D_-F(x)$ for a.e. x.

$G(x) := -F(-x)$

$$D^+G(x) = \lim_{\delta\to 0^+} \sup_{h\in(0,\delta)} \frac{G(x+h)-G(x)}{h}$$
$$= \lim_{\delta\to 0^+} \sup_{h\in(0,\delta)} \frac{+(F(-x-h)-F(-x))}{-h}$$
$$= \lim_{\delta\to 0^+} \sup_{k\in(-\delta,0)} \frac{F(-x+k)-F(-x)}{k} \qquad \boxed{k=-h}$$
$$= D^-F(-x) \qquad \forall x\in[-b,-a]$$

$D_-G(x) = D_+F(-x)$

If (ii) true $\Rightarrow$ $D^+G(x) \le D_-G(x)$ $\forall x\in[-b,-a]$ a.e.

$D^-F(x) \le D_+F(x)$ $\forall$ a.e. $x\in[a,b]$.

$\overset{+\text{(ii)}}{\Longrightarrow}$ $D^+F \le D_-F \le D^-F \le D_+F \le D^+F < \infty$

Proof of (i):

$E_N := \{x\in[a,b] : D^+F(x) > N\}$

$\bigcap_{N=1}^{\infty} E_N = \{x : D^+F(x) = \infty\}$.

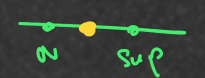
$G(x) := F(x) - Nx$.

$G(b) - G(a) = (F(b)-F(a)) - N(b-a)$.

$\boxed{\text{Claim:}}$ $E_N \subset \{x\in[a,b] : G(y) > G(x)\ \exists y > x\} = \coprod_{i=1}^{\infty}(c_i, d_i)$

$x\in E_N \Rightarrow D^+F(x) > N \Rightarrow \inf_{\delta>0} \sup_{h\in(0,\delta)} \frac{F(x+h)-F(x)}{h} > N$

$\Rightarrow \forall\delta>0,\ \sup_{h\in(0,\delta)} \frac{F(x+h)-F(x)}{h} > N$

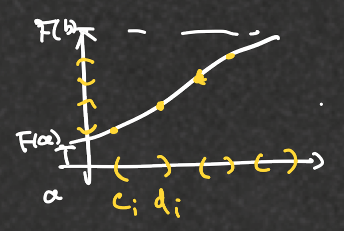


$\Rightarrow \exists h\in(0,\delta)$ s.t. $\frac{F(x+h)-F(x)}{(x+h)-x} > N$

$\Rightarrow G(x+h) - G(x) > 0$. ∎

$$\mu(E_N) \le \mu\left(\coprod_{i=1}^{\infty}(c_i,d_i)\right) = \sum_{i=1}^{\infty}(d_i - c_i) = \sum_{i=1}^{\infty}\frac{1}{N}(F(d_i)-F(c_i))$$

$G(d_i) = G(c_i) \Rightarrow F(d_i) - Nd_i = F(c_i) - Nc_i$



$\mu(E_N) \le \frac{1}{N}(F(b)-F(a))$.

$\lim_{N\to\infty}\mu(E_N) = 0$.

$\mu\left(\bigcap_{N=1}^{\infty} E_N\right)$

$\mu\{x : D^+F(x) = \infty\}$ ∎(i).

$A := \{x\in[a,b] : D^+F(x) > D_-F(x)\}$

Ask: $\mu(A) \overset{?}{=} 0$.

$A_{R,r} := \{x\in[a,b] : D^+F(x) > R > r > D_-F(x)\}$.
$(R>r)$, $\in\mathbb{Q}$

WANT: $\mu(A_{R,r}) = 0$ $\forall R > r$.

$A = \bigcup_{\substack{R>r\\(R,r)\in\mathbb{Q}^2}} A_{R,r}$.

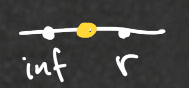
$\forall\varepsilon>0$, $\exists\, \mathcal{O} = \coprod_{n=1}^{\infty} I_n$ s.t. $\mu(\mathcal{O}) < \mu(A_{R,r}) + \varepsilon$.

I_n : Consider $J_n := \{x\in I_n : H(y) < H(x)\ \exists y < x\} = \coprod_{k=1}^{\infty}(c_{n,k}, d_{n,k})$
$I_n = (a_n, b_n)$; $-H(y) > -H(x)$

$H(x) = F(x) - rx$

If $\{r > D_-F(x)\}$, $x\in I_n$, then $r > \sup_{\delta\to 0}\inf_{h\in(-\delta,0)} \frac{F(x+h)-F(x)}{h}$

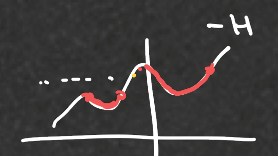
$\Rightarrow r > \inf_{h\in(-\delta,0)} \frac{F(x+h)-F(x)}{h}$ $\forall\delta$.



$\Rightarrow \exists h\in(-\delta,0)$ s.t. $\frac{F(x+h)-F(x)}{h} < r$

$H(x+h) < H(x)$.

$\Rightarrow x\in J_n$



Similar argument as before: $H(c_{n,k}) = H(d_{n,k})$
$\iff F(c_{n,k}) - F(d_{n,k}) = r(c_{n,k} - d_{n,k})$

$\{x : r > D_-F(x)\}\cap I_n \subset J_n = \coprod_{k=1}^{\infty}(c_{n,k}, d_{n,k})$

On each $(c_{n,k}, d_{n,k})$:

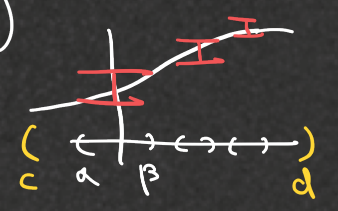
$U_{n,k} := \{x\in(c_{n,k}, d_{n,k}) : D^+F(x) > R\}$.
$\subset \{x\in(c_{n,k},d_{n,k}) : G(y) > G(x)\ \exists y > x\}$.
$G(x) = F(x) - Rx$
$= \coprod_{l=1}^{\infty}(\alpha_{n,k,l}, \beta_{n,k,l})$

$\underbrace{\quad\ \ }_{A_{R,r}} \subset \bigcup_n I_n$

$F(\alpha_{n,k,l}) - F(\beta_{n,k,l}) = R(\alpha_{n,k,l} - \beta_{n,k,l})$

$$A_{R,r} = \coprod_n (A_{R,r}\cap I_n) \subset \coprod_n \{x\in[a,b] : D^+F(x) > R\}\cap J_n$$
$$= \coprod_n \coprod_{k=1}^{\infty} \{x\in[a,b] : D^+F(x) > R\}\cup(c_{n,k}, d_{n,k})$$
$$= \coprod_{n=1}^{\infty}\coprod_{k=1}^{\infty} U_{n,k} \subset \coprod_{n=1}^{\infty}\coprod_{k=1}^{\infty}\coprod_{l=1}^{\infty}(\alpha_{n,k,l}, \beta_{n,k,l})$$

$$\mu(A_{R,r}) \le \sum_{n,k,l=1}^{\infty}(\beta_{n,k,l} - \alpha_{n,k,l})$$
$$= \frac{1}{R}\sum_{n,k,l=1}^{\infty}(F(\beta_{n,k,l}) - F(\alpha_{n,k,l}))$$
$$\le \frac{1}{R}\sum_{n,k=1}^{\infty}(F(d_{n,k}) - F(c_{n,k}))$$
$$= \frac{r}{R}\sum_{n,k=1}^{\infty}(d_{n,k} - c_{n,k})$$
$$\le \frac{r}{R}\sum_{n=1}^{\infty}\mu(I_n) = \frac{r}{R}\mu(\mathcal{O}) \le \frac{r}{R}(\mu(A_{R,r}) + \varepsilon)$$



let $\varepsilon\to 0^+$ $\Rightarrow$ $\underbrace{\mu(A_{R,r})}_{<\infty} \le \underbrace{\frac{r}{R}}_{\in(0,1)}\underbrace{\mu(A_{R,r})}_{<\infty}$

$\Rightarrow \mu(A_{R,r}) = 0$.

$$\mu\{x : D^+F(x) > D_-F(x)\} = \mu\left(\bigcup_{\substack{R>r>0\\(R,r)\in\mathbb{Q}^2}} A_{R,r}\right) = 0$$

∎(ii)