

Hardy-Littlewood's maximal
function:

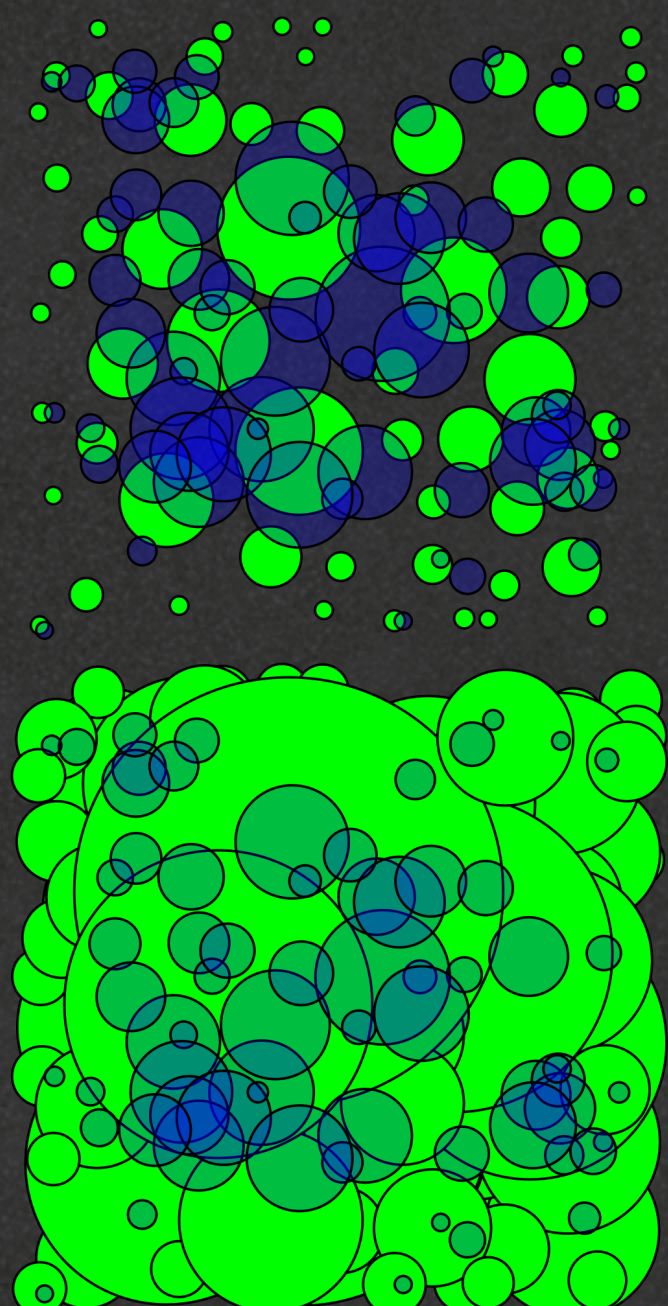
$$Mf(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B f d\mu$$

Lebesgue Differentiation Theorem

$f \in L^1(\mathbb{R}^d)$, then

$$\lim_{\substack{\mu(B) \rightarrow 0 \\ B \ni x}} \frac{1}{\mu(B)} \int_B f d\mu = f(x) \quad \underline{\text{a.e.}}$$

Vitali's 3-covering



$\Rightarrow$

Hardy-Littlewood's inequality

$$\mu\{x: Mf(x) > \alpha\} \leq \frac{3^d}{\alpha} \|f\|_{L^1}$$

Tchebychev's inequality

$$\mu\{x: |f(x)| > \alpha\} \leq \frac{1}{\alpha} \|f\|_{L^1}$$

$$\overline{C_c(\mathbb{R}^d)}^{L^1} = L^1(\mathbb{R}^d)$$



Locally integrable

$$Mf(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B f d\mu$$

~~$f: \mathbb{R} \rightarrow \mathbb{R}$
 $f(x) = \frac{1}{x}$~~

f is said to be locally L^1 ($f \in L^1_{loc}(\mathbb{R}^d)$)

if $f \chi_B$ is L^1 $\forall B$ ball in $\mathbb{R}^d$.

Lebesgue Differentiation Theorem

$$f \in L^1_{loc}(\mathbb{R}^d) \rightarrow \lim_{\mu(B) \rightarrow 0, B \ni x} \frac{1}{\mu(B)} \int_B f d\mu = f(x) \text{ a.e.}$$

HW: $f \chi_{B_N}$



e.g. $f(x) = e^{|x|} \notin L^1(\mathbb{R}^d)$
 $\in L^1_{loc}(\mathbb{R}^d)$

E measurable set $\subset \mathbb{R}^d$.

If $\lim_{\mu(B) \rightarrow 0, B \ni x} \frac{\mu(E \cap B)}{\mu(B)} = 1$, then say



x is a point of $\wedge$ density (Lebesgue.)



Claim: Almost all points $x \in E$ are points of density.

Proof: $f = \chi_E \in L^1_{loc}(\mathbb{R}^d)$. Leb. Diff. Theorem

$$\Rightarrow \lim_{\mu(B) \rightarrow 0, B \ni x} \frac{1}{\mu(B)} \int_B \chi_E d\mu = \chi_E(x) \text{ a.e. } x$$

$$\Rightarrow \lim_{\mu(B) \rightarrow 0, B \ni x} \frac{1}{\mu(B)} \mu(E \cap B) = 1 \text{ a.e. } x \in E.$$

$$\frac{d}{dr} \frac{\mu(M \cap B_r)}{\mu(B_r)}$$



$$\lim_{\mu(B) \rightarrow 0, B \ni x} \frac{1}{\mu(B)} \int_B |f(y) - f(x)| dy = 0 \text{ a.e. } x.$$

$$L.D.T. \Rightarrow \left| \frac{1}{\mu(B)} \int_B (f(y) - f(x)) dy \right| \leq \frac{1}{\mu(B)} \int_B |f(y) - f(x)| dy$$

$$|f(y) - f(x)| \leq |f(y) - r| + |r - f(x)|, r \in \mathbb{Q}$$

$$L.D.T. \Rightarrow \lim_{\mu(B) \rightarrow 0, B \ni x} \frac{1}{\mu(B)} \int_B |f(y) - r| dy = \underbrace{|f(x) - r|}_{g_r(x)} \text{ a.e. } x \in \mathbb{R}^d.$$

$$\text{Suppose } \lim_{\mu(B) \rightarrow 0, B \ni x} \frac{1}{\mu(B)} \int_B |f(y) - r| dy = \underbrace{|f(x) - r|}_{g_r(x)} \cdot \chi_{E_r} \text{ where } \mu(E_r) = 0.$$

Consider $E = \bigcup_{r \in \mathbb{Q}} E_r$, $x \notin E$, consider $f(x)$
 $\mu = 0$

density of $\mathbb{Q} \Rightarrow \forall \varepsilon > 0, \exists r \in \mathbb{Q}$ s.t. $|f(x) - r| < \varepsilon/2$

$$\frac{1}{\mu(B)} \int_B |f(y) - f(x)| d\mu \leq \frac{1}{\mu(B)} \int_B |f(y) - r| d\mu + |f(x) - r|$$

$$\lim_{\mu(B) \rightarrow 0, B \ni x} \frac{1}{\mu(B)} \int_B |f(y) - f(x)| d\mu \leq |f(x) - r| + |f(x) - r| < \varepsilon$$

$$\Rightarrow \lim_{\mu(B) \rightarrow 0, B \ni x} \frac{1}{\mu(B)} \int_B |f(y) - f(x)| d\mu = 0 \quad \forall x \notin E$$

□

$$f \in L^p(\mathbb{R}^d) \quad (p > 1) \quad \|f\|_p < +\infty.$$

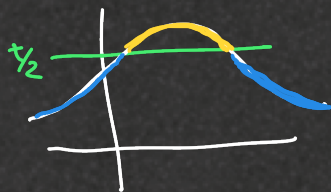
$$\|Mf\|_p \leq C_{d,p} \|f\|_p \quad \leftarrow \text{strong type estimate.}$$

Proof $\underbrace{Mf(x)}_{=} \leq \|f\|_\infty$

$$\sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f| d\mu$$

HW: $\|Mf\|_p^p = \int_{\mathbb{R}^d} |Mf|^p d\mu \stackrel{\text{Fubini-Tonelli}}{=} \int_0^\infty p t^{p-1} \mu\{x: Mf(x) > t\} dt$

$$f = \underbrace{f \chi_{\{|f| > \frac{t}{2}\}}}_{\text{yellow}} + \underbrace{f \chi_{\{|f| \leq \frac{t}{2}\}}}_{\text{blue}}$$



$$Mf(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f| d\mu$$

$$\begin{aligned} &\leq \sup_{B \ni x} \frac{1}{\mu(B)} \int_B \underbrace{|f \chi_{\{|f| > \frac{t}{2}\}}|}_{\text{yellow}} + \sup_{B \ni x} \frac{1}{\mu(B)} \int_B \underbrace{|f \chi_{\{|f| \leq \frac{t}{2}\}}|}_{\text{blue}} \\ &\leq M(f \chi_{\{|f| > \frac{t}{2}\}})(x) + \frac{t}{2} \leq \frac{t}{2} \end{aligned}$$

$$x \in \{Mf > t\} \Rightarrow M(f \chi_{\{|f| > \frac{t}{2}\}}) > \frac{t}{2}$$

$$\Rightarrow \{x: Mf(x) > t\} \subset \{x: M(f \chi_{\{|f| > \frac{t}{2}\}}) > \frac{t}{2}\}$$

$$\|Mf\|_p^p = \int_0^\infty p t^{p-1} \mu\{x: Mf(x) > t\} dt$$

$$\leq \int_0^\infty p t^{p-1} \mu\{x: M(f \chi_{\{|f| > \frac{t}{2}\}}) > \frac{t}{2}\} dt$$

$$\stackrel{HL}{\leq} \int_0^\infty p t^{p-1} \cdot \frac{3^d}{t/2} \int_{\mathbb{R}^d} |f \chi_{\{|f| > \frac{t}{2}\}}| dx dt$$

$$= \int_{\mathbb{R}^d} \int_0^\infty p t^{p-1} \cdot \frac{3^d \cdot 2}{t} |f| \chi_{\{|f| > \frac{t}{2}\}} dt dx$$

$$= \int_{\mathbb{R}^d} |f| p \cdot 3^d \cdot 2 \left(\int_0^\infty t^{p-2} \chi_{\{|f| > \frac{t}{2}\}} dt \right) dx$$

$$= \int_{\mathbb{R}^d} |f(x)| \cdot 2p 3^d \left(\int_0^{|f(x)|} t^{p-2} dt \right) dx$$

$\chi_{[0, |f(x)|]}^{(t)}$

$$= \int_{\mathbb{R}^d} |f(x)| \cdot 2p 3^d \cdot \frac{1}{p-1} |f(x)|^{p-1} dx$$

$$= \left(\frac{2p \cdot 3^d}{p-1} \right) \|f\|_p^p$$

Remark: Stein: can be replaced by C_p .

Zorn's Lemma

$(X, \leq)$:

$$\begin{aligned}
 (1) \quad & x \leq x \\
 (2) \quad & x \leq y \text{ and } y \leq x \Rightarrow x = y \\
 (3) \quad & x \leq y \text{ and } y \leq z \Rightarrow x \leq z.
 \end{aligned}$$
 then call $(X, \leq)$ is partially ordered set.

e.g. $X =$ set of all subspaces of $\mathbb{R}^d$.
 $u \leq v \stackrel{\text{def}}{\iff} u \subset v$.

chain: $Y \subset X$ if any pair $y_1, y_2 \in Y$ is either $y_1 \leq y_2$ or $y_2 \leq y_1$.
 $\uparrow$ chain

upper bound: Y is a chain in $(X, \leq)$.
 $u \in X$ is an upper bound of Y if $y \leq u \quad \forall y \in Y$.

maximal element: $m \in X$ is called a maximal element if $m \leq x$, then $m = x$.

Zorn's lemma: If any chain $\mathcal{C}$ in X has an upper bound, then X has a maximal element.

Kin-Yin Li's 4063.

W vector space over $\mathbb{R}$.

$X := \{S \mid S \text{ is a linearly indep. subset of } W\}$

$\mathcal{C} = \{S_\alpha\}_{\alpha \in A}$ chain. $\bigcup_{\alpha \in A} S_\alpha$ is an upper bound.

Zorn's lemma $\Rightarrow \exists S_M \in X$ maximal element.

WANT: $\text{span } S_M = W$.

If not, $\exists x \in W$ s.t. $x \notin \text{span } S_M$.

$\underbrace{S_M \cup \{x\}}_{\text{linearly indep.}} \supset S_M \Rightarrow S_M \cup \{x\} = S_M$.


Vitali's 5-covering lemma.

$\mathcal{F}$ = collection of balls in $\mathbb{R}^d$.

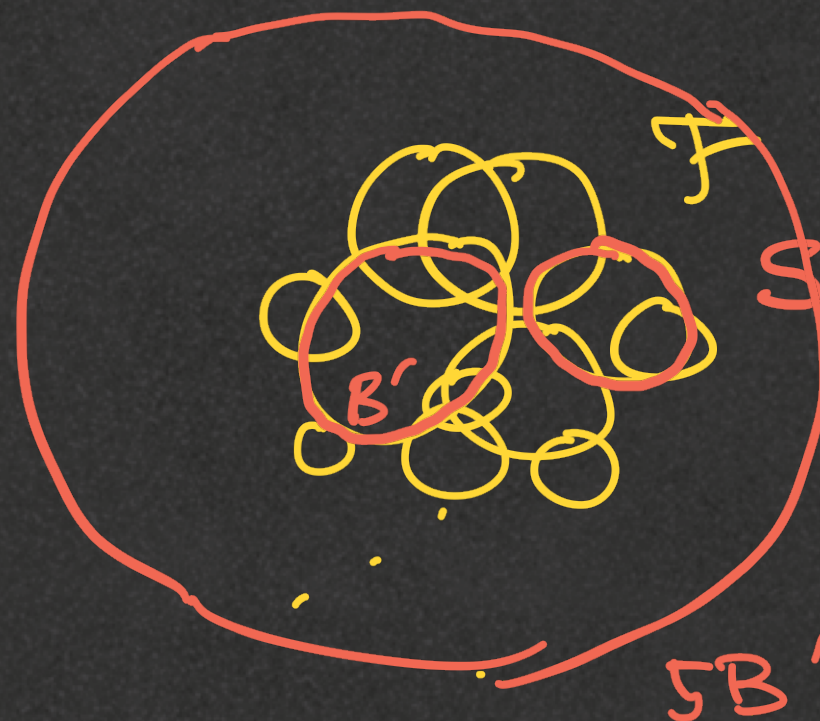
and $\sup_{B \in \mathcal{F}} \{\text{radius}(B)\} < +\infty$.

then: $\exists \mathcal{G} \subset \mathcal{F}$ st. $\mathcal{G}$ consists of disjoint balls

and $\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B' \in \mathcal{G}} 5B'$.

Proof (Outline):

$X := \left\{ \underline{S} \subset \mathcal{F} : \begin{array}{l} \text{for any } B \in \mathcal{F} \text{ that intersects } B' \in S \\ \text{has property that } 5B' \supset B \end{array} \right\}$



$S_1 \preceq S_2 \stackrel{\text{def}}{\iff} S_1 \subseteq S_2$.

$\mathcal{C}$ chain in X . $\leadsto$ upper bound
 $u := \bigcup_{S \in \mathcal{C}} S$.

Zorn's lemma $\Rightarrow \exists$ maximal element $S_n \in X$ (HW).