Zorn's Lemma

Correct version:

"If every chain $\mathcal{C}$ in a partially ordered set X has an upper bound in X, then X has a maximal element."

Example: Prove any vector space has a basis.

Let $X = \{ S : S \text{ is a set of linearly indep. vectors in } W \}$.

$\mathcal{C}$ chain in X. $\mathcal{C} = \{ S_\alpha \}$

$\bigcup_{S_\alpha \in \mathcal{C}} S_\alpha \leftarrow$ upper bound?

$\cup$

$S_\alpha \ \forall S_\alpha \in \mathcal{C}$

$v_1, \dots, v_n \in \bigcup_{S_\alpha \in \mathcal{C}} S_\alpha$

let $v_i \in S_{\alpha_i}$

WLOG assume S_{α_n} is largest.

$\underbrace{c_1 v_1 + \dots + c_n v_n = 0}$ $\rightsquigarrow$

$v_1, \dots, v_n \in S_{\alpha_n} \leftarrow$ linear indep. $\Rightarrow c_i = 0$. $\Rightarrow \{v_1, \dots, v_n\}$ lin. indep.

$\Rightarrow \bigcup_{S_\alpha \in \mathcal{C}} S_\alpha \in X$.

Zorn's lemma $\Rightarrow \exists$ maximal element $x_M \in X$.

Claim: x_M is a basis for W.

Proof: Suppose $\exists w_0 \in W$ s.t. $\operatorname{span}\{x_M\} \not\ni w_0$

Argue $\{w_0\} \cup x_M \in X$ $\Rightarrow\Leftarrow$.

Fundamental Theorem of Calculus

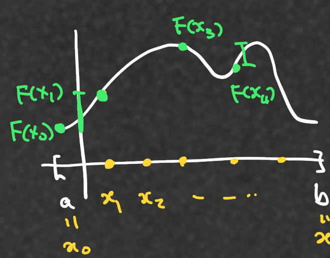
$$F(b) - F(a) = \int_a^b F'(x)\,dx$$

Still true for Lebesgue integrals for measurable F?

① Total bounded variation

② absolute continuity

$F: \mathbb{R} \to \mathbb{R}$ $\qquad T_F(a,b) \qquad (V_a^b(f))$



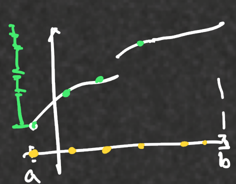
$$:= \sup_P \sum_{i=1}^n |F(x_i) - F(x_{i-1})|$$

P = partitions of $[a,b]$.

F is said to have <u>bounded total variations</u> on $[a,b]$

$\overset{def}{\iff} T_F(a,b) < +\infty$.

① Example: $F: [a,b] \to \mathbb{R}$ monotone $\nearrow$.



$\forall \{x_i\}$:

$$\sum_{i=1}^n |F(x_i) - F(x_{i-1})|$$
$$= \sum_{i=1}^n (F(x_i) - F(x_{i-1}))$$
$$= F(x_n) - F(x_0) = F(b) - F(a) < \infty.$$

$T_F(a,b) = F(b) - F(a) < +\infty$.

② $F: [a,b] \to \mathbb{R}$ differentiable, and $|F'| \le M$.

$\forall \{x_i\}$, $\quad \sum_{i=1}^n |F(x_i) - F(x_{i-1})| = \sum_{i=1}^n |F'(\underset{(x_{i-1},x_i)}{\underset{\frown}{\xi_i}})|\,|x_i - x_{i-1}|$

$\le M \sum_{i=1}^n |x_i - x_{i-1}| = M(b-a) < +\infty$.

$T_F(a,b) \le M(b-a) < \infty$.

③ $F(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad : [0,1] \to \mathbb{R}$.

$\frac{1}{x} = \frac{\pi}{2} + 2k\pi$

$\{x_i\} = \{0, \frac{1}{\frac{3\pi}{2}+2k\pi}, \frac{1}{\frac{\pi}{2}+2k\pi}, \ldots, \frac{1}{\frac{3\pi}{2}+2\pi}, \frac{1}{\frac{\pi}{2}+2\pi}, \frac{1}{\frac{3\pi}{2}}, \frac{1}{\frac{\pi}{2}}, 1\}$

$\qquad -1 \quad 1 \quad -1 \quad 1$

$\sum |F(x_i) - F(x_{i-1})|$

$\ge 2(2k)$

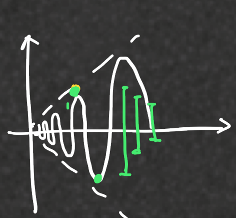
$\sup_P \sum_i |F(x_i) - F(x_{i-1})| \ge 4k \quad \forall k \in \mathbb{N}$.

$\Rightarrow T_F(0,1) = +\infty$.

④ $F(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad : [0,1] \to \mathbb{R}$

$|F'(x)| \le 3 \rightsquigarrow T_F(0,1) \le 3(1-0) = 3$.

⑤ $F(x) = \begin{cases} x \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad : [0,1] \to \mathbb{R}$.



<u>HW</u>: $F(x) = \begin{cases} x^a \sin\frac{1}{x^b} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad : [0,1] \to \mathbb{R}$.

Show F has total bounded variation on $[0,1]$

$\iff b < a$.

F bounded variation on $[a,b]$.

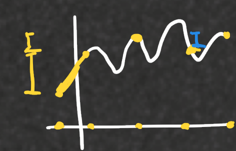
<u>Claim</u>: $F = g - h$ where $g, h \nearrow$ on $[a,b]$.

<u>Key idea</u>: $\underset{\text{variable}}{[a,\overset{\uparrow}{x}]} \subset [a,b]$

$T_F(a,x) = \sup_P \sum |F(t_i) - F(t_{i-1})| \qquad a = t_0 < t_1 < \cdots < t_n = x$.

$P_F(a,x) = \sup_P \sum_{\{i: F(t_i) \ge F(t_{i-1})\}} (F(t_i) - F(t_{i-1})) \qquad$ (positive variation)

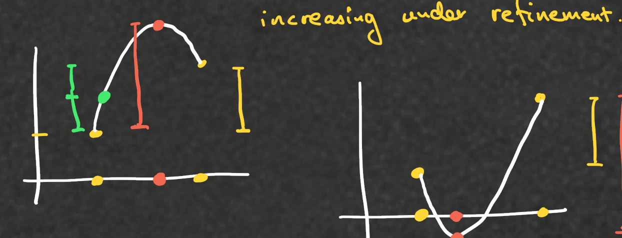
$N_F(a,x) = \sup_P \sum_{\{i: F(t_i) \le F(t_{i-1})\}} -(F(t_i) - F(t_{i-1}))$



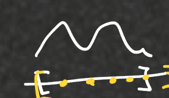
<u>Goal</u>: $F(x) = F(a) + P_F(a,x) - N_F(a,x)$

• $P_F(a,x) = \sup_P \left(\sum_{(+)} |F(t_i) - F(t_{i-1})| \right)$

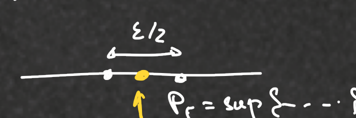
increasing under refinement.



• $P_F(a,x) \nearrow$, $N_F(a,x) \nearrow$ as $x \nearrow$.



$\forall \varepsilon > 0$, <u>Need</u>: $|F(x) - F(a) - P_F(a,x) + N_F(a,x)| < \varepsilon$.



$\forall \varepsilon > 0$, $\exists P = \{t_i\}$ (part. of $[a,x]$) s.t.

$\left| P_F(a,x) - \left(\sum_{(+)} (F(t_i) - F(t_{i-1})) \right) \right| < \frac{\varepsilon}{2}$

$\exists Q = \{s_j\}$ s.t.

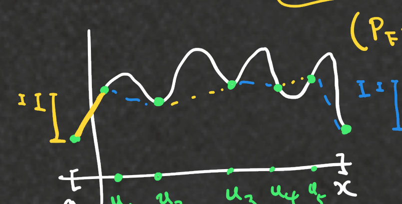
$\left| N_F(a,x) - \sum_{(-)} (-(F(s_j) - F(s_{j-1}))) \right| < \frac{\varepsilon}{2}$

$P \cup Q = \{u_k\}$ partition of $[a,x]$, s.t.

$\left| P_F(a,x) - \sum_{(+)} (F(u_k) - F(u_{k-1})) \right| < \varepsilon/2$

$\left| N_F(a,x) - \sum_{(-)} -(F(u_k) - F(u_{k-1})) \right| < \varepsilon/2$

$$F(x) - F(a) = \underbrace{\sum_{(+)} (F(u_k) - F(u_{k-1}))}_{(P_F - \varepsilon/2,\ P_F + \varepsilon/2)} - \underbrace{\sum_{(-)} -(F(u_k) - F(u_{k-1}))}_{\underbrace{F(u_{k-1}) - F(u_k)}_{(N_F - \varepsilon/2,\ N_F + \varepsilon/2)}}$$



$F(u_1) - F(a)$
$+ \cancel{F(u_3)} - \cancel{F(u_2)}$
$+ \cancel{F(u_5)} - \cancel{F(u_4)}$

$-$

$\begin{pmatrix} \cancel{F(u_1)} - \cancel{F(u_2)} \\ + \cancel{F(u_3)} - \cancel{F(u_4)} \\ + \cancel{F(u_5)} - F(x) \end{pmatrix} = F(x) - F(a)$.

$|F(x) - F(a) - (P_F - N_F)| < \varepsilon$.

$\varepsilon \to 0 \Rightarrow F(x) = F(a) + P_F(a,x) - N_F(a,x)$

• $T_F(a,x) = \sup - \quad - \quad -$

$\shortparallel$

$P_F(a,x) + N_F(a,x)$

$\sup\{\ \cdot\ - \}$

$= \sup\{\quad\} + \sup\{- \ldots\}$