① Generalize **Fundamental Theorem of Calculus** to Lebesgue integrals.

$$\boxed{\begin{array}{c} f:\mathbb{R}\to\mathbb{R} \\ \dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x) \end{array}} \quad \overset{?}{\rightsquigarrow}$$

$$f(x) = \frac{d}{dx}\int_{[a,x]} f(t)\,dt\Big|_x = \lim_{h\to 0} \frac{\int_{[a,x+h]} f(t)\,dt - \int_{[a,x]} f(t)\,dt}{h}$$

$$= \lim_{h\to 0} \frac{\int_{[x,x+h]} f(t)\,dt}{h} = \lim_{h\to 0} \frac{1}{\mu([x,x+h])} \int_{[x,x+h]} f(t)\,dt$$

$f: \mathbb{R}^d \to \mathbb{R}$

$$\lim_{r\to 0} \frac{1}{\mu(B)} \int_B f\,d\mu \overset{?}{=} f(x), \qquad B = B(x,r)$$

Hardy-Littlewood's maximal function.

$f: \mathbb{R}^d \to \mathbb{R}$, $L^1(\mathbb{R}^d)$

B ← open

$Mf: \mathbb{R}^d \to \mathbb{R}$

$$\underset{\mathbb{R}^d}{\underset{\cap}{Mf(x)}} = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f|\,d\mu$$

Hardy-Littlewood's maximal function.

Given $f : \mathbb{R}^d \to \mathbb{R}$, $L^1(\mathbb{R}^d)$.

(i) $Mf : \mathbb{R}^d \to \mathbb{R}$ is measurable

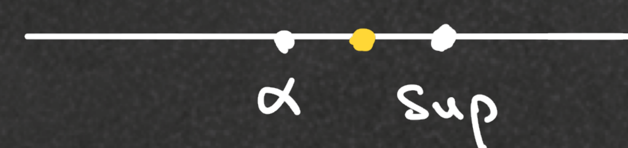
(ii) $Mf(x) < \infty$ for a.e. x

(iii) $\left\{ \mu \{x \in \mathbb{R}^d : Mf(x) > \underset{\alpha \in \mathbb{R}_+}{\alpha} \} \le \frac{3^d}{\alpha} \|f\|_{L^1} \right\}$

Hardy-Littlewood inequality.

<u>(i)</u>

$$Mf(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f| \, d\mu.$$



$(Mf)^{-1}((\alpha, +\infty))$ is <u>measurable</u>?

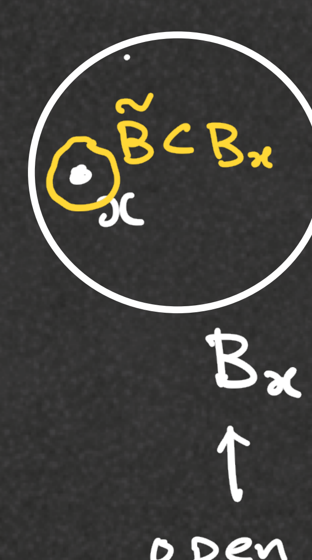
<u>Claim</u>: $(Mf)^{-1}((\alpha, +\infty))$ is open. $\forall \alpha \in \mathbb{R}$

<u>Proof</u>: $x \in (Mf)^{-1}((\alpha, +\infty))$

$$\Leftrightarrow (Mf)(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f| \, d\mu. > \alpha$$

$\Leftrightarrow \exists B_x \ni x$ s.t.

$$Mf(x) \ge \frac{1}{\mu(B_x)} \int_{B_x} |f| > \alpha$$



$$\underset{\subset B_x}{y \in \tilde{B}} \Rightarrow Mf(y) = \sup_{B \ni y} \frac{1}{\mu(B)} \int_B |f| \, d\mu$$

$$\ge \frac{1}{\mu(B_x)} \int_{B_x} |f| \, d\mu > \alpha$$

$$\Rightarrow \tilde{B} \subset (Mf)^{-1}((\alpha, +\infty))$$

$\blacksquare$

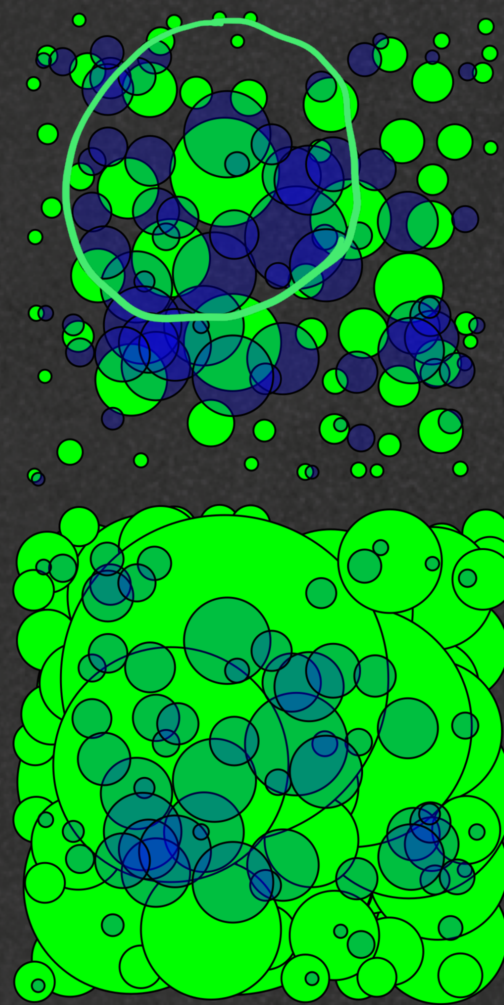
Vitali 3-covering lemma:

Given: $\{B_1, \dots, B_N\} \subset \mathbb{R}^d$

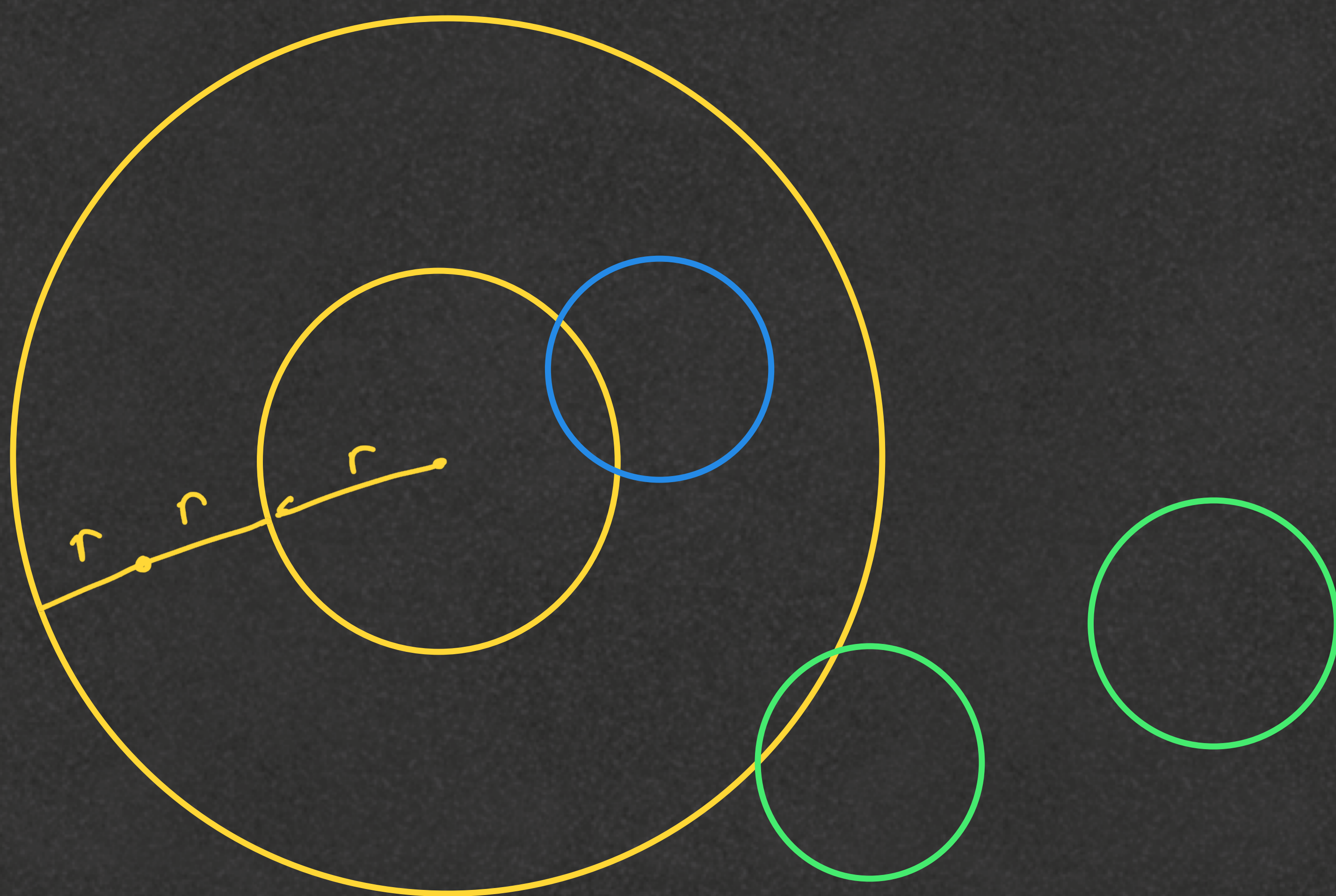
<u>then</u>: $\exists \underbrace{\{B_{i_1}, \dots, B_{i_n}\}}_{\text{disjoint}} \subset \{B_1, \dots, B_N\}$

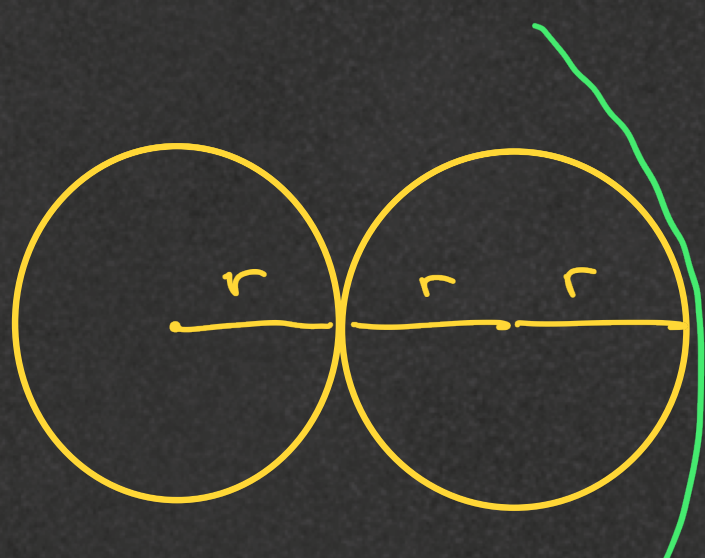
s.t. $\bigcup_{j=1}^{n} \underbrace{3B_{i_j}}_{=} \supset \bigcup_{i=1}^{N} B_i$

$$\begin{cases} B = B(x, r) \\ 3B = B(x, 3r). \end{cases}$$



Key idea of proof:





Tchebychev:

$f : \mathbb{R}^d \to \overline{\mathbb{R}}$, $L^1(\mathbb{R}^d)$

$$\mu\{x \in \mathbb{R}^d : |f(x)| > \underset{\underset{\mathbb{R}_+}{\uparrow}}{\alpha}\} \le \frac{1}{\alpha} \int_{\mathbb{R}^d} |f| \, d\mu = \frac{1}{\alpha} \|f\|_{L^1}.$$

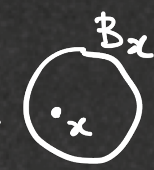
Proof: $$\int_{\mathbb{R}^d} \underbrace{|f|}_{\geq 0} d\mu \geq \int_{\{|f(x)| > \alpha\}} \overset{>\alpha}{|f|} d\mu \geq \alpha \, \mu\{x : |f(x)| > \alpha\}. \quad \blacksquare$$

Hardy-Littlewood's inequality.

$f : \mathbb{R}^d \to \mathbb{R}$, $L^1(\mathbb{R}^d)$

WANT: $$\mu\{\underbrace{x \in \mathbb{R}^d : Mf(x) > \alpha}_{E_\alpha}\} \le \frac{3^d}{\alpha} \|f\|_{L^1}.$$

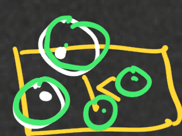
$$x \in E_\alpha \Rightarrow \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f| \, d\mu > \alpha$$



$$\Rightarrow \exists B_x \ni x \quad \text{s.t.} \quad \frac{1}{\mu(B_x)} \int_{B_x} |f| \, d\mu > \alpha.$$

Claim: K compact $\subset E_\alpha$. want: $\mu(K) \le \frac{3^d}{\alpha} \|f\|_{L^1}$

Proof $$K \subset \bigcup_{x \in K} B_x \overset{\text{cpt}}{\Rightarrow} \exists x_1, \dots, x_n \in K$$
s.t. $$K \subset \bigcup_{i=1}^n B_{x_i}$$



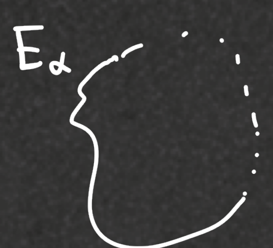
$$\mu(K) \le \mu\left(\bigcup_{i=1}^n B_{x_i}\right)$$

$$\{B_{x_1}, \dots, B_{x_n}\} \supset^{\exists} \{B_{x_{j_1}}, \dots, B_{x_{j_m}}\} \leftarrow \text{disjoint}$$ (Vitali)

s.t. $$\bigcup_{i=1}^m 3B_{x_{j_i}} \supset \bigcup_{i=1}^n B_{x_i}$$

$$\mu(K) \le \mu\left(\bigcup_{i=1}^m 3B_{x_{j_i}}\right) \le \sum_{i=1}^m \mu(3B_{x_{j_i}}) = \sum_{i=1}^m 3^d \mu(B_{x_{j_i}})$$

$$\le 3^d \cdot \sum_{i=1}^m \frac{1}{\alpha} \int_{B_{x_{j_i}}} |f| \, d\mu$$ disjoint



$$= \frac{3^d}{\alpha} \int_{\coprod_{i=1}^m B_{x_{j_i}}} |f| \, d\mu \le \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f| \, d\mu.$$

independent of K.

$$E_\alpha = \bigcup_{N=1}^\infty \big(\underbrace{[-N,N]^n \cap E_\alpha}_{\text{bounded.}}\big)$$

$$\subset \bigcup_{N=1}^\infty \underbrace{\left(\overline{[-N,N]^n \cap E_\alpha}\right)}_{K_N} \leftarrow \text{monotone}$$

$$\Rightarrow \mu(E_\alpha) \le \mu\left(\bigcup_{N=1}^\infty K_N\right)$$

$$= \lim_{N \to \infty} \underbrace{\mu(K_N)}_{\le 3^d/\alpha \|f\|_{L^1}} \le \frac{3^d}{\alpha} \|f\|_{L^1}.$$

Lebesgue Differentiation Theorem:

$f \in L^1(\mathbb{R}^d)$, then:

$$\left[\lim_{\substack{\mu(B)\to 0\\ B\ni x}} \frac{1}{\mu(B)} \int_B f \, d\mu = f(x)\right] \quad \forall \text{ a.e. } x \in \mathbb{R}^d$$

$$\forall \varepsilon > 0, \ \exists \delta > 0 \quad \text{s.t.} \quad \begin{cases}\mu(B) < \delta \\ B \ni x\end{cases} \Rightarrow \left|\frac{1}{\mu(B)}\int_B f\,d\mu - f(x)\right| < \varepsilon.$$

Leb. Diff. Theorem is "obviously" true for continuous functions.

g continuous, (in particular at x) $\Rightarrow \forall \varepsilon > 0$, $\exists \delta > 0$,
$|y - x| < \delta \Rightarrow |g(y) - g(x)| < \varepsilon$.

$$\frac{1}{\mu(B)}\int_B g(y)\,dy - g(x) = \frac{1}{\mu(B)}\int_B g(y)\,dy - \frac{1}{\mu(B)}\int_B g(x)\,dy$$

$y \in B(x, \delta)$

$$= \frac{1}{\mu(B)}\int_B \big(g(y) - g(x)\big)\,dy.$$

$f \in L^1(\mathbb{R}^d)$,

$\forall \varepsilon > 0$, $\exists g_\varepsilon : \mathbb{R}^d \to \mathbb{R}$, continuous, compact support

s.t. $\|f - g_\varepsilon\|_{L^1} < \varepsilon$.

$$\lim_{\substack{\mu(B)\to 0\\ B\ni x}} \frac{1}{\mu(B)}\int_B g_\varepsilon\,d\mu = g_\varepsilon(x) \quad \text{for all } x \in \mathbb{R}^d.$$

$\boxed{\alpha > 0}$

$$E_\alpha := \left\{x : \limsup_{\substack{\delta\to 0^+ \ \mu(B)<\delta\\ B\ni x}} \left|\frac{1}{\mu(B)}\int_B f(y)\,dy - f(x)\right| > 2\alpha\right\}$$

$$\left|\frac{1}{\mu(B)}\int_B f(y)\,dy - f(x)\right| \le \frac{1}{\mu(B)}\int_B |f(y) - g(y)|\,dy$$
$$+ \left|\frac{1}{\mu(B)}\int_B \big(g(y) - g(x)\big)\,dy\right|$$
$$+ |g(x) - f(x)|$$

$$\limsup_{\substack{\mu(B)\to 0\\ B\ni x}} \left|\frac{1}{\mu(B)}\int_B f(y)\,dy - f(x)\right| \le \limsup_{\substack{\mu(B)\to 0\\ B\ni x}} \frac{1}{\mu(B)}\int_B |f(y) - g(y)|\,dy$$
$$+ 0 + |g(x) - f(x)|.$$

$$\le M(f-g)(x) + |f(x) - g(x)|.$$

$x \in E_\alpha \Rightarrow 2\alpha < M(f-g)(x) + |f(x) - g(x)|$.

$\Rightarrow M(f-g)(x) > \alpha$ OR $|f(x) - g(x)| > \alpha$.

$\Rightarrow x \in \{x : M(f-g) > \alpha\} \cup \{|f(x) - g(x)| > \alpha\}$.

$\mu\{x : M(f-g) > \alpha\} \le \frac{3^d}{\alpha}\|f - g\|_{L^1}$ (Hardy-Littlewood)

$\mu\{x : |f(x) - g(x)| < \alpha\} \le \frac{1}{\alpha}\|f - g\|_{L^1}$. (Tcheysehv..)

$E_\alpha \subset \{ \quad \} \cup \{ \quad \} \Rightarrow \mu(E_\alpha) \le \left(\frac{3^d}{\alpha} + \frac{1}{\alpha}\right)\|f - g\|_{L^1}$

indep. of ε.

$$< \left(\frac{3^d}{\alpha} + \frac{1}{\alpha}\right)\varepsilon.$$

true for any ε.

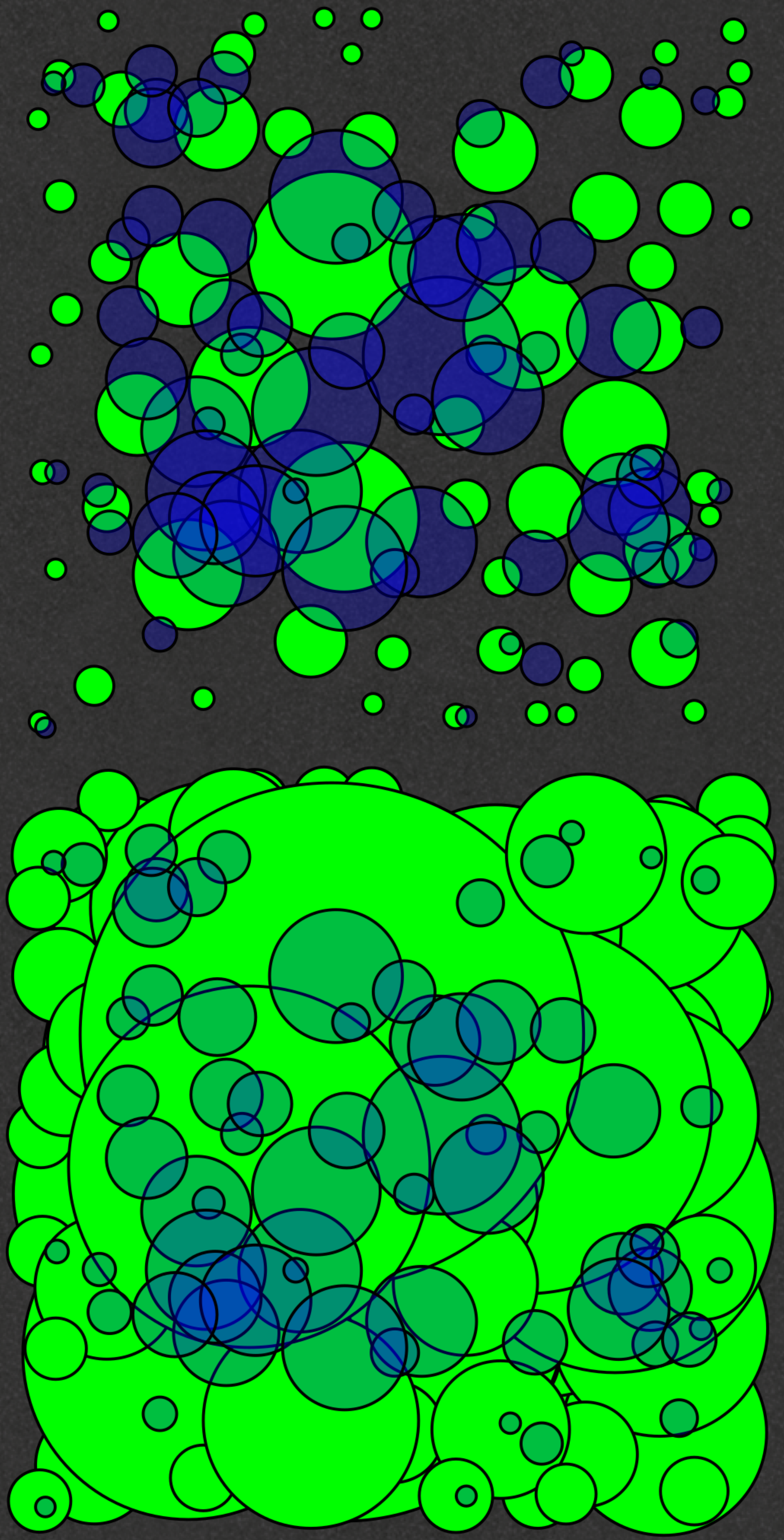
let $\varepsilon \to 0^+ \Rightarrow \mu(E_\alpha) = 0$

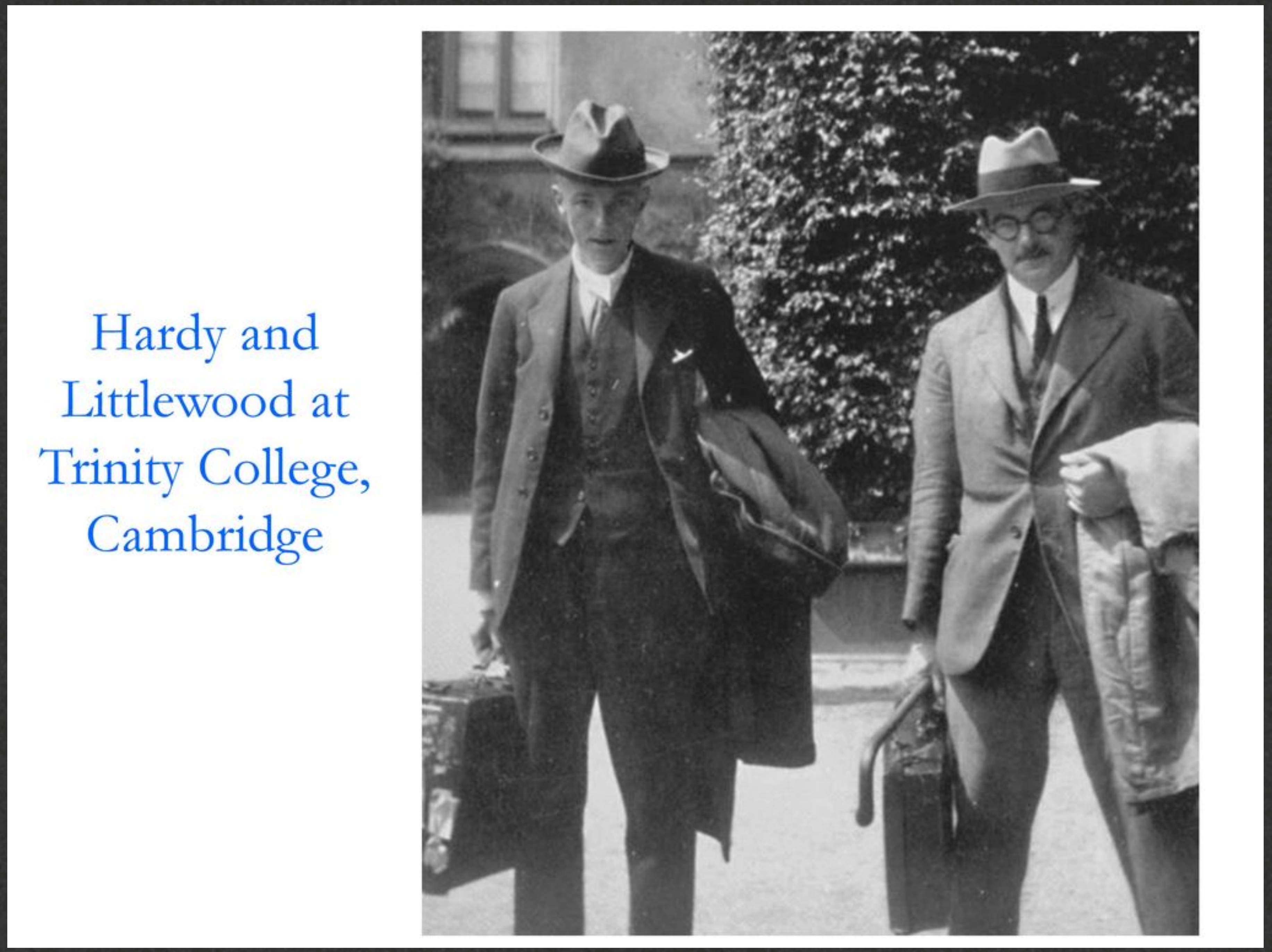
$$E := \left\{x \in \mathbb{R}^d : \limsup_{\substack{\mu(B)\to 0\\ B\ni x}} \left|\frac{1}{\mu(B)}\int_B f(y)\,dy - f(x)\right| > 0\right\}$$

$$= \bigcup_{n=1}^{\infty} E_{\frac{1}{n}} = \left\{x : | - \ - \ - | > \tfrac{2}{n}\right\}$$

$\mu = 0$

$\Rightarrow \mu(E) = 0 \quad \frac{1}{\mu(B)}\int_B f(y)\,dy \to f(x) \quad \forall x \text{ a.e. in } \mathbb{R}^d.$ $\blacksquare$




Hardy and
Littlewood at
Trinity College,
Cambridge