Tonelli's Theorem:

$f \geq 0$ measurable on $\mathbb{R}^{d_1+d_2}$.

$$\int_{\mathbb{R}^{d_1+d_2}} f \, dx\,dy = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f^y \, dx \right) dy$$

(hold true even if $\infty = \infty$.)

(Fubini requires $\int_{\mathbb{R}^{d_1+d_2}} |f| \, dx\,dy < \infty$).

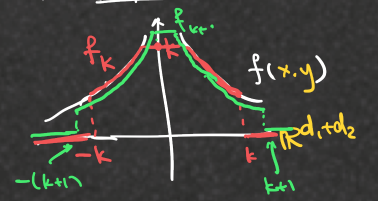
$\int_{\mathbb{R}^{d_1+d_2}} f \, dx\,dy$

To use Fubini's Theorem in practice:

Tonelli's Theorem

$$\Rightarrow \int_{\mathbb{R}^{d_1+d_2}} |f| \, dx\,dy = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} |f|^y \, dx \right) dy. \checkmark$$

Proof of Tonelli



For each $k \in \mathbb{N}$ define f_k as in the diagram.

$f_{k+1}(x,y) \geq f_k(x,y)$.

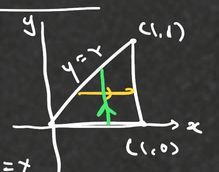
$f_k \in L^1(\mathbb{R}^{d_1+d_2}) \ \forall k$. $\quad f_k \to f$

(i) ✓
(ii) ✓
(iii)

$$\forall k: \int_{\mathbb{R}^{d_1+d_2}} f_k(x,y) \, dx\,dy = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f_k^y(x) \, dx \right) dy.$$

$\lim \int$ (MCT) $\lim \int \int$

$$\int_{y=0}^{y=1} \int_{x=y}^{x=1} \frac{\sin x}{x} \, dx\,dy$$



$$\int_{x=0}^{x=1} \int_{y=0}^{y=x} \frac{\sin x}{x} \, dy\,dx = \int_{x=0}^{x=1} \left[\frac{\sin x}{x} y \right]_{y=0}^{y=x} dx$$

$\frac{\sin x}{x} \geq 0$.

$$= \int_{x=0}^{x=1} \frac{\sin x}{\cancel{x}} \cdot \cancel{x} \, dx = [-\cos x]_0^1 = \ < \infty$$

Tonelli's $\Rightarrow \int_{x=0}^{x=1} \int_{y=0}^{y=x} \frac{\sin x}{x} \, dy\,dx < \infty$

$$= \int_{\triangle} \frac{\sin x}{x} \, dA < \infty.$$

Fubini

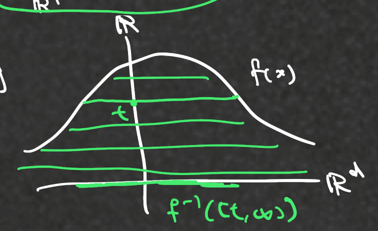
$$\int_{\triangle} \frac{\sin x}{x} \, dA = \int_?^? \int_?^? \frac{\sin x}{x} \, dx\,dy$$

Prop: $f \geq 0 : \mathbb{R}^d \to \mathbb{R}$

Claim: $\int_{\mathbb{R}^d} f \, d\mu = \int_0^\infty \mu_{\mathbb{R}^d} \{ x \in \mathbb{R}^d : f(x) \geq t \} \, dt$

$f^{-1}([t,\infty))$

Proof: $A = \{(x,t) : 0 \leq t \leq f(x)\}$



$$\int_{\mathbb{R}^{d+1}} \chi_A = \int_0^\infty \left(\int_{\mathbb{R}^d} \chi_A^t(x) \, dx \right) dt$$

$\chi_A(x,t)$

$$= \int_0^\infty \left(\int_{\mathbb{R}^d} \chi_{\{x : f(x) \geq t\}}(x) \, dx \right) dt$$

$$= \int_0^\infty \boxed{\mu\{x : f(x) \geq t\}} \, dt$$

$t \geq 0$

$\chi_A^t(x) = \chi_A(x,t)$ $\quad (x,t) \in A$

$$= \begin{cases} 1 & \text{if } 0 \leq t \leq f(x) \\ 0 & \text{otherwise} \end{cases}$$

$$= \chi_{\{x : f(x) \geq t\}}(x)$$

$$\int_{\mathbb{R}^{d+1}} \chi_A(x,t) = \int_{\mathbb{R}^d} \left(\int_0^\infty \chi_A^x(t) \, dt \right) dx$$

$$= \int_{\mathbb{R}^d} \left(\int_0^\infty \chi_{[0,f(x)]}(t) \, dt \right) dx$$

$$\chi_A^x(t) = \begin{cases} 1 & 0 \leq t \leq f(x) \\ 0 & \text{otherwise} \end{cases}$$

$$= \chi_{[0,f(x)]}(t).$$

$$= \int_{\mathbb{R}^d} \underbrace{\mu[0,f(x)]}_{f(x)} \, dx = \int_{\mathbb{R}^d} f \, dx.$$

$S = [0,1] \times N$

(x, y), $N \subset [0,1]$

Claim: S is not measurable.

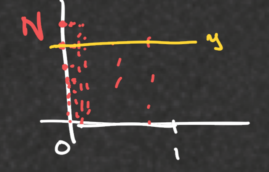
Proof: Suppose otherwise S is measurable.

$\Rightarrow \chi_S$ is measurable function, integrable.

Fubini – Tonelli's

$\Rightarrow \exists A \subset [0,1]$, $\mu(A) = 0$ s.t. χ_S^y is measurable $\forall y \notin A$.

$\exists B \subset [0,1]$, $\mu(B) = 0$ s.t. $\boxed{\chi_S^x \text{ is measurable. } x \notin B}$



$$\chi_S^y(x) \equiv \begin{cases} 1 & \text{if } y \in N \\ 0 & \text{if } y \notin N \end{cases}$$

$$\chi_S^x(y) = \begin{cases} 1 & \text{if } y \in N \\ 0 & \text{if } y \notin N \end{cases} = \boxed{\chi_N(y)}$$

Good news: $E_1 \subset \mathbb{R}^{d_1}$, $E_2 \subset \mathbb{R}^{d_2}$ both measurable

$\Rightarrow E_1 \times E_2 \subset \mathbb{R}^{d_1+d_2}$ also measurable.

Proof: $\exists G_i \in G_\delta(\mathbb{R}^{d_i})$ s.t. $\mu(G_i) = \mu(E_i)$.
$G_i \supset E_i$ $\quad (i = 1, 2)$.

Consider $G = G_1 \times G_2 \supset E_1 \times E_2$.

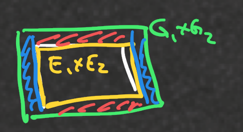
$$G_1 = \bigcap_{i=1}^\infty O_i \qquad G_1 \times G_2 = \bigcap_{i,j=1}^\infty (O_i \times \tilde{O}_j) \text{ is } G_\delta(\mathbb{R}^{d_1+d_2})$$

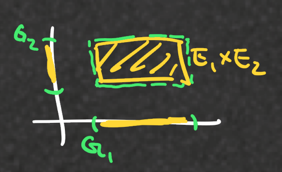
$$G_2 = \bigcap_{j=1}^\infty \tilde{O}_j$$

$(x,y) \in G_1 \times G_2 \Leftrightarrow \begin{cases} x \in O_i \ \forall i \\ y \in \tilde{O}_j \ \forall j \end{cases} \Leftrightarrow (x,y) \in O_i \times \hat{O}_j \ \forall i,j$

$G_1 \times G_2 - E_1 \times E_2$

$$= \underbrace{((G_1 - E_1) \times G_2)}_{\mu=0} \cup (E_1 \times \underbrace{(G_2 - E_2)}_{\mu=0})$$





Fubini $\Rightarrow \mu(G_1 \times G_2 - \underbrace{E_1 \times E_2}) = 0$.

$\therefore$ measurable.