L^1-space

$\|f\|_{L^1} := \int_{\mathbb{R}^d} |f|\,d\mu$ Leb

Define $f = g$ if $f = g$ a.e.

$\mathcal{L}^1 = \{f : \int_{\mathbb{R}^d} |f|\,d\mu < \infty\}$

$f \sim g \overset{\text{def}}{\iff} f = g$ a.e.

$L^1 = \mathcal{L}^1/\sim$

Theorem

$L^1(X)$ is a complete normed vector space (Banach space).

Proof: Given $\{f_n\}$ L-Cauchy (i.e. $\forall \varepsilon > 0$, $\exists N$ s.t. $m, n \geq N \Rightarrow \|f_m - f_n\|_{L^1} < \varepsilon$)

Cauchy $\Rightarrow \exists n_1 < n_2 < n_3 < \cdots$

s.t. $\|f_{n_{k+1}} - f_{n_k}\|_{L^1} < \frac{1}{2^k}$

$f_{n_k}(x) := f_{n_1}(x) + \sum_{j=1}^{k-1} (f_{n_{j+1}} - f_{n_j})(x)$

Claim: $\sum_{j=1}^{\infty}$ converges for a.e. x.

$g(x) := |f_{n_1}(x)| + \sum_{j=1}^{\infty} |f_{n_{j+1}}(x) - f_{n_j}(x)| = \begin{cases} \infty \\ \text{finite} \end{cases}$

$\int_X g(x)\,d\mu = \int_X |f_{n_1}(x)| + \int_X \sum_{j=1}^{\infty} |f_{n_{j+1}} - f_{n_j}|\,d\mu$

$\overset{(MCT)}{=} \underbrace{\int_X |f_{n_1}|}_{\|f_{n_1}\|_{L^1} < \infty} + \sum_{j=1}^{\infty} \underbrace{\int_X |f_{n_{j+1}} - f_{n_j}|\,d\mu}_{\|f_{n_{j+1}} - f_{n_j}\|_{L^1} \leq \frac{1}{2^j}} < \infty$

$\mu\{x : g(x) = \infty\} = 0$

$g(x) = \text{finite}$ a.e. on $X \Rightarrow \exists f_{n_k} \to f$ a.e. on X

Next: $f \in L^1(X)$ and $\|f_n - f\|_{L^1} \to 0$.

$\int_X |f| = \int_X |f_{n_1} + \sum_{j=1}^{\infty} (f_{n_{j+1}} - f_{n_j})|$

$\leq \int_X |f_{n_1}| + \sum_{j=1}^{\infty} |f_{n_{j+1}} - f_{n_j}|$

$= \underbrace{\int_X |f_{n_1}|}_{< \infty} + \underbrace{\sum_{j=1}^{\infty} \int_X |f_{n_{j+1}} - f_{n_j}|}_{\leq \sum_{j=1}^{\infty} \frac{1}{2^j}} < \infty$

$\Rightarrow f \in L^1(X)$

$|f_{n_k} - f| = \left| f_{n_1} + \sum_{j=1}^{k-1} (f_{n_{j+1}} - f_{n_j}) - f_{n_1} - \sum_{j=1}^{\infty} (f_{n_{j+1}} - f_{n_j}) \right| = \left| \sum_{j=k}^{\infty} (f_{n_{j+1}} - f_{n_j}) \right|$

$\leq \sum_{j \geq k} |f_{n_{j+1}} - f_{n_j}|$

$\leq g$

$\int g\,d\mu < \infty$.

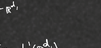
LDCT $\lim_{k \to \infty} \int_X |f_{n_k} - f|\,d\mu = \int_X \underbrace{\lim_{k \to \infty} |f_{n_k} - f|}_{= 0 \text{ a.e.}}\,d\mu = 0$.

$f \in L^1(X)$

$\forall \varepsilon > 0$, $\exists \varphi \in C_c^\infty$ s.t. $\int_X |f - \varphi|\,d\mu < \varepsilon$ $\Big\}$

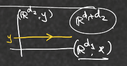
$\|f - \varphi\|_{L^1} < \varepsilon$.

$\overline{C_c^\infty} = L^1$



$\overline{\mathbb{Q}} = \mathbb{R}$.

Fubini's Theorem



$f : \mathbb{R}^{d_1+d_2} \to \mathbb{R}$

$\int_{\mathbb{R}^{d_1+d_2}} f(x,y) \overset{?}{=} \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)\,dx \right) dy$.

$\overset{?}{=}$

$\int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)\,dy \right) dx$

$\boxed{\int_{\mathbb{R}^{d_1+d_2}} |f|\,d\mu < \infty}$

Given $f \in L^1(\mathbb{R}^{d_1+d_2})$

then $\exists A \subset \mathbb{R}^{d_2}$ s.t. $\mu(A) = 0$ s.t.

$y \notin A \Rightarrow$ (i) $f^y(x) := f(x,y) : \mathbb{R}^{d_1} \to \mathbb{R}$ is in $L^1(\mathbb{R}^{d_1})$

(ii) $g(y) := \int_{\mathbb{R}^{d_1}} f^y(x)\,dx$ is $L^1(\mathbb{R}^{d_2})$.

(iii) $\int_{\mathbb{R}^{d_1+d_2}} f(x,y)\,dx\,dy = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)\,dx \right) dy$

$\mathcal{F} := \{ f \in L^1(\mathbb{R}^{d_1+d_2}) : f$ satisfies (i), (ii), (iii) $\}$.

Strein's trick: show $\mathcal{F} \supset L^1(\mathbb{R}^{d_1+d_2})$.

Step 1: $\{f_1, \ldots, f_n\} \subset \mathcal{F} \Rightarrow \boxed{\sum_{j=1}^{n} c_j f_j} \in \mathcal{F}$.

$c_j \in \{1, \ldots, n\}$

f_j^y is $L^1(\mathbb{R}^{d_1})$

$y \notin A = \bigcup_{k=1}^{n} A_k$, $\mu(A_k) = 0$.

$\sum_{i=1}^{n} c_i f_i^y \in L^1(\mathbb{R}^{d_1})$ (i) ✓

$g(y) = \int_{\mathbb{R}^{d_1}} \sum_{i=1}^{n} c_i f_i^y(x)\,dx = \sum_{i=1}^{n} c_i \underbrace{\int_{\mathbb{R}^{d_1}} f_i^y(x)\,dx}_{f_i \in \mathcal{F} \quad L^1(\mathbb{R}^{d_1})}$

$\in L^1(\mathbb{R}^{d_2})$ (ii)

$\int_{\mathbb{R}^{d_1+d_2}} \sum_{i=1}^{n} c_i f_i(x,y)\,dx = \sum_{i=1}^{n} c_i \underbrace{\int_{\mathbb{R}^{d_1+d_2}} f_i(x,y)\,dx\,dy}_{\in \mathcal{F}}$

$= \cdots = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \sum_i c_i f_i^y(x)\,dx \right) dy$.

Step 2: Given $f \in L^1(\mathbb{R}^{d_1+d_2})$, $\exists \{f_k\} \subset \mathcal{F}$ s.t. $f_k \to f$ a.e. $\forall k \in \mathbb{N}$.

$f \in L^1(\mathbb{R}^{d_1+d_2})$

$f \in L^1(\mathbb{R}^{d_1+d_2})$

Claim: $f \in \mathcal{F}$.

Proof of Claim:

(i) $\forall k \in \mathbb{N}$, $f_k \in \mathcal{F} \Rightarrow \exists A_k \subset \mathbb{R}^{d_2}$ s.t. $\mu(A_k) = 0$ and $\forall y \notin A_k$, $f_k^y \in L^1(\mathbb{R}^{d_1})$.

Take $A := \bigcup_{k=1}^{\infty} A_k$. $\mu(A) = 0$.

Consider $y \notin A \Rightarrow y \notin A_k \ \forall k$.

$\lim_{k \to \infty} \int_{\mathbb{R}^{d_1}} f_k^y(x)\,dx \overset{(MCT)}{=} \int_{\mathbb{R}^{d_1}} \lim_{k \to \infty} f_k^y(x)\,dx = \int_{\mathbb{R}^{d_1}} f^y(x)\,dx$

$g(y)$ $\qquad f(x,y)$ $\qquad \boxed{g(y)}$

$\int_{\mathbb{R}^{d_2}} g(y)\,dy = \int_{\mathbb{R}^{d_2}} \lim_{k \to \infty} g_k(y)\,dy$

$\overset{(MCT)}{=} \lim_{k \to \infty} \int_{\mathbb{R}^{d_2}} g_k(y)\,dy$

$= \lim_{k \to \infty} \int_{\mathbb{R}^{d_2}} \underbrace{\left(\int_{\mathbb{R}^{d_1}} f_k^y(x)\,dx \right)}_{g_k(y)} dy$

$= \lim_{k \to \infty} \int_{\mathbb{R}^{d_1+d_2}} f_k(x,y)\,dx\,dy$.

$\overset{(MCT)}{=} \int_{\mathbb{R}^{d_1+d_2}} \lim_{k \to \infty} f_k(x,y)\,dx\,dy$

$= \int_{\mathbb{R}^{d_1+d_2}} f(x,y)\,dx\,dy < \infty$.

$\Rightarrow \mu\{y : g(y) = \infty\} = 0$ — (i) ✓

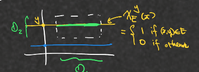
(ii) ✓

$\int_{\mathbb{R}^{d_1+d_2}} f(x,y)\,dx\,dy = \cdots = \int_{\mathbb{R}^{d_2}} \underbrace{\left(\int_{\mathbb{R}^{d_1}} f(x,y)\,dx \right)}_{g(y)} dy$ (iii) ✓

Step 3: Verify: $\chi_E \in \mathcal{F}$ $\forall E \in G_\delta$ (countable intersection of open sets)

(a) E = open cube

$= \underbrace{\mathcal{O}_1}_{\subset \mathbb{R}^{d_1}} \times \underbrace{\mathcal{O}_2}_{\subset \mathbb{R}^{d_2}}$



$\chi_E^y(x) = \begin{cases} 1 & \text{if } (x,y) \in E \\ 0 & \text{if otherwise} \end{cases}$

$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \chi_E^y(x)\,dx \right) dy$

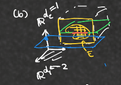
$= \int_{\mathcal{O}_2} \left(\int_{\mathbb{R}^{d_1}} \chi_E^y(x)\,dx \right) dy$

$= \int_{\mathcal{O}_2} \left(\int_{\mathbb{R}^{d_1}} \chi_{\mathcal{O}_1}(x)\,dx \right) dy$

$= \int_{\mathcal{O}_2} \left(\mu_{\mathbb{R}^{d_1}}(\mathcal{O}_1) \right) dy = \boxed{\mu_{\mathbb{R}^{d_1}}(\mathcal{O}_1)\,\mu_{\mathbb{R}^{d_2}}(\mathcal{O}_2)}$

$= \int_{\mathcal{O}_1 \times \mathcal{O}_2} \chi_E(x,y)\,dx\,dy$.

(b)



Claim: $\chi_E \in \mathcal{F}$

$\chi_E^y(x) = \begin{cases} 0 \\ 1 \end{cases}$

$\Rightarrow \int_{\mathbb{R}^{d_1+d_2}} \chi_E(x,y) = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \chi_E^y(x)\,dx \right) dy = 0$. ✓

(c) U = open set $\subset \mathbb{R}^{d_1+d_2}$

Claim: $\chi_U \in \mathcal{F}$.

U = union of almost disjoint rectangular cubes.

$U = \bigcup_{n=1}^{\infty} \square_n \cup \bigcup_{k=1}^{\infty} \text{faces}$

$\chi_U = \sum_{n=1}^{\infty} \underbrace{\chi_{\square_n}}_{\in \mathcal{F}} + \sum_{k=1}^{\infty} \underbrace{\chi_{\text{face}}}_{\in \mathcal{F}} \in \mathcal{F}$.

(d) $E \in G_\delta$. $\boxed{\chi_E \overset{?}{\in} \mathcal{F}}$

$\bigcap_{n=1}^{\infty} \mathcal{O}_n$

$\tilde{\mathcal{O}}_1 = \mathcal{O}_1 \cap (\mathcal{O}_1 \cap \mathcal{O}_2) \cap (\mathcal{O}_1 \cap \mathcal{O}_2 \cap \mathcal{O}_3)$

$\tilde{\mathcal{O}}_2$ $\qquad \tilde{\mathcal{O}}_3$

$\tilde{\mathcal{O}}_1 \supset \tilde{\mathcal{O}}_2 \supset \tilde{\mathcal{O}}_3 \supset \cdots$ $\qquad E = \bigcap_{n=1}^{\infty} \tilde{\mathcal{O}}_n$

$\chi_E = \lim_{n \to \infty} \chi_{\mathcal{O}_n} \in \mathcal{F}$

$\overset{?}{\in} \mathcal{F}$

Step 4: $\mu(E) = 0 \rightsquigarrow \chi_E \in \mathcal{F}$

measurable

$\int_{\mathbb{R}^{d_1+d_2}} \chi_E = \mu(E) = 0$.

$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \chi_E^y \right) dx\,dy = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \chi_E^y \right) dx\,dy$.

$\exists G \in G_\delta$ s.t. $\underset{0}{\mu(E)} = \mu(G)$ $\qquad \chi_E^y = \chi_G^y$ a.e. y.

$\boxed{E}$

Round stuff

Step 5: E measurable $\subset \mathbb{R}^{d_1+d_2}$

$\exists \underset{E}{\hat{G}} \in G_\delta$ s.t. $\mu(E) = \mu(G)$ $\qquad \boxed{E}$ $\boxed{G}$

$E = G - (G - E)$

$\chi_G = \chi_E + \underbrace{\chi_{G-E}}_{\mu = 0}$

Step 6: $f = f_+ - f_-$

$f_+ = \lim_{n \to \infty} \underbrace{\varphi_n}_{\text{simple}} \in \mathcal{F}$ $\qquad \in \mathcal{F}$

$\square$