MATH 3043 • Fall 2019 • Honors Analysis II
Problem Set #3 • Due Date: 10/11/2019, 11:59PM

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1. (10 points) In Royden's book, the Lebesgue integral for a bounded measurable function $f : X \to \mathbb{R}$, where $\mu(X) < +\infty$, is given by the following. First he defined the Lebesgue integral for simple functions (as in Stein's book). Then, he defined that f is Lebesgue integrable when $\inf\left\{\int_X \psi\, d\mu : \psi \text{ is simple and } f \leq \psi\right\} = \sup\left\{\int_X \phi\, d\mu : \varphi \text{ is simple and } \varphi \leq f\right\}$. If it happens, the Lebesgue integral $\int_X f\, d\mu$ is defined as either one of the above values.

   Show that for a measurable function f, Royden's definition gives the same integral as both Stein's and Yan's definitions (just show one, as we have already shown Stein's and Yan's definitions are the same).

2. (12 points) Let (X, Σ) be a space with σ-algebra Σ, and μ and ν are two measures defined on (X, Σ) such that $\mu(A) \leq \nu(A) < +\infty$ for any $A \in \Sigma$. Given a non-negative measurable function $f : X \to [0, \infty)$, show that

$$\int_X f\, d\mu \leq \int_X f\, d\nu$$

   using two different definitions of Lebesgue integrals (a) Stein's definition; (b) Yan's definition

3. (15 points) Suppose $(\mathbb{R}^n, \Sigma, \mu)$ is a measure space such that Σ contains all Borel sets in $\mathbb{R}^n$. Given an uncountable family of continuous function $\{f_\alpha : \mathbb{R}^n \to \mathbb{R}\}_{\alpha \in \mathcal{A}}$, we define

$$F(x) := \sup\{f_\alpha(x) : \alpha \in \mathcal{A}\}.$$

   (a) Show that F is a measurable function.

   (b) Suppose f_α is non-negative and Lebesgue integrable for any $\alpha \in \mathcal{A}$. Is F always Lebesgue integrable? If yes, prove it; if not, give a counterexample.

4. (15 points) Given a set X equipped with an outer measure μ^*. Suppose $\{f_n : X \to \mathbb{R}\}_{n=1}^{\infty}$ is a sequence of measurable functions, such that it converges to a function $f : X \to \mathbb{R}$ in measure, i.e. for any $\varepsilon > 0$, we have

$$\lim_{n \to \infty} \mu^*\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\} = 0.$$

   Show that f is measurable. Note that we do NOT assume a priori that f is measurable, so be careful when applying results which require measurability.

5. (10 points) Consider a double-indexed sequence $\{a_{i,j}\}_{i,j \in \mathbb{N}}$ of real numbers. Suppose there exists a sequence $\{b_i\}_{i=1}^{\infty}$ such that $|a_{i,j}| \leq b_i$ for any $i, j \in \mathbb{N}$ and $\sum_{i=1}^{\infty} b_i < +\infty$. Show that

$$\lim_{j \to +\infty} \sum_{i=1}^{\infty} a_{i,j} = \sum_{i=1}^{\infty} \lim_{j \to +\infty} a_{i,j}.$$

6. (15 points) Complete the proof of the first step of Proposition 10.5.3, i.e. show that for any $\varepsilon > 0$, and any bounded interval (a, b), there exists $g \in C_c^{\infty}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \left|\chi_{(a,b)} - g\right|\, d\mu < \varepsilon.$$

   [Note: Just quote why such your function is C^∞, no need to check it again. Hint: first define a family of smooth functions with several parameters, then show using intermediate-value theorem that some of the parameters could work.]