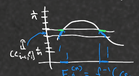
Stein $\quad \mu(X)<\infty \quad |f|\le M \quad$ Min γ_{∞}

$$\cancel{\int_X} f\,d\mu = \lim_{n\to\infty}\int_X \varphi_n\,d\mu \qquad (\varphi_n \to f)$$

$\forall n\in\mathbb{N},\ \exists\, P^{(n)} = \bigcup_{i=1}^{\infty} X_i^{(n)}$ s.t.

$$Q \ge P^{(n)} \Rightarrow \left|\cancel{\int} f\,d\mu - S(Q, \{x_i^*\}, f)\right| \le \frac{1}{n}.$$



$\varphi^{(n)} := \sum_i \sup_{E_i^{(n)}} f\ \chi_{E_i^{(n)}} \xrightarrow{n\to\infty} f$

$E_i^{(n)} = f^{-1}(\alpha_{i-1}, \alpha_i]$ $\quad$ Ex: $|\varphi^{(n)} - f| \le \frac{1}{n}$.

$$Q^{(n)} = \left\{\bigcup_{i,j} E_i^{(n)} \cap X_j^{(n)}\right\}$$

$$\left|\cancel{\int} \varphi^{(n)}\,d\mu - S(Q, \{y_j^*\}, f)\right|$$

$$= \left|\sum_{i,j} \sup_{E_i^{(n)}} f\ \mu(E_i^{(n)}) - \sum_{i,j} f(y_j^*)\,\mu(E_i^{(n)}\cap X_j^{(n)})\right|$$

$\mu(E_i^{(n)}) = \sum_j \mu(E_i^{(n)}\cap X_j^{(n)})$

$$= \left|\sum_{i,j} \underbrace{\left(\sup_{E_i^{(n)}} f - f(y_j^*)\right)}_{\le \frac{1}{n}} \mu(E_i^{(n)}\cap X_j^{(n)})\right| \le \frac{1}{n}\underbrace{\mu(X)}_{\to 0}.$$

$$\Rightarrow \left|\int_X \varphi^{(n)}\,d\mu - \cancel{\int_X} f\,d\mu\right| \le \frac{1}{n}(\mu(X)+1) \to 0.$$

$\downarrow n\to\infty$

$$\left|\cancel{\int_X} f\,d\mu - \cancel{\int_X} f\,d\mu\right| = 0$$

- Example:

$$\lim_{n\to\infty}\int_{[0,1]} \frac{nx\log x}{1+n^2x^2}\,d\mu(x)$$

- $|f_n| \le g$ a.e. indep. of n.
- $\int_X g\,d\mu < \infty$

$$\left|\frac{nx\log x}{1+n^2x^2}\right| \qquad \frac{y}{1+y^2} \le \frac{1}{2} \qquad \Rightarrow \lim_X f_n = \int_X \lim f_n$$

$\Leftrightarrow 1+y^2 \ge 2y$

$\le \frac{1}{2}\underbrace{|\log x|}_{g(x)}$ on $[0,1]$ $\quad \Leftrightarrow (1-y)^2 \ge 0$. $\quad$ LDCT

$= -\frac{1}{2}\log x$

$$\int_{[0,1]} -\frac{1}{2}\log x\,dx = \int_0^1 -\frac{1}{2}\log x\,dx = -\frac{1}{2}\left[x\log x\right]_0^1 + \frac{1}{2}\int_0^1 1\,dx = 0 + \frac{1}{2} < \infty.$$

LDCT

$$\Rightarrow \lim_{n\to\infty}\int_{[0,1]} \frac{nx\log x}{1+n^2x^2}\,dx = \int_{[0,1]} \lim_{n\to\infty}\frac{nx\log x}{1+n^2x^2}\,dx = \int_{[0,1]} 0\,dx = 0$$

- Borel-Cantelli:

$$\begin{cases} \{E_n\} \subset X, & \sum_{n=1}^{\infty}\mu(E_n) < \infty \\ S = \bigcap_{n=1}^{\infty}\bigcup_{k\ge n} E_k = \{x\in X : x \text{ is } \infty\text{-many } E_n\text{'s}\}. \end{cases}$$

$\Rightarrow \mu(S) = 0$.

Proof: $\quad f = \sum_{n=1}^{\infty} \chi_{E_n}$

$f = +\infty$ on S. If $x\in S$, $\Rightarrow f(x) = +\infty$.

$$\int_S f\,d\mu \le \int_X f\,d\mu = \int_X \sum_{n=1}^{\infty}\chi_{E_n}\,d\mu = \int_X \lim_{N\to\infty}\sum_{n=1}^{N}\underbrace{\chi_{E_n}}_{\ge 0}$$

as $N\nearrow\infty$.

$$= \lim_{N\to\infty}\sum_{n=1}^{N}\int_X \chi_{E_n}$$

$$= \sum_{n=1}^{\infty}\mu(E_n) < \infty \quad \leftarrow \text{assumption.}$$

- Example:

$$\begin{cases} f(x,t): \mathbb{R}\times\mathbb{R}\to\mathbb{R}, \text{ is measurable for each fixed } t. \\ \frac{\partial f}{\partial t} \text{ exist and bounded.} \end{cases}$$

$$\Rightarrow \frac{d}{dt}\int_a^b f(x,t)\,dx = \int_a^b \frac{\partial f}{\partial t}(x,t)\,dx.$$

$\leftarrow$ finite $\quad \uparrow$ finite

Proof:

$$\frac{d}{dt}\int_a^b f(x,t)\Big|_{t=t_0} = \lim_{t\to t_0}\frac{\int_a^b f(x,t)\,dx - \int_a^b f(x,t_0)\,dx}{t-t_0}$$

$$= \lim_{t\to t_0}\int_a^b \frac{1}{t-t_0}(f(x,t)-f(x,t_0))\,dx$$

Let $\{t_n\} \to t_0$ as $n\to\infty$.

$$\lim_{n\to\infty}\int_a^b \underbrace{\frac{1}{t_n-t_0}(f(x,t_n)-f(x,t_0))}_{F_n(x)}\,dx$$

$$|F_n(x)| = \frac{|f(x,t_n)-f(x,t_0)|}{|t_n-t_0|} = \left|\frac{\partial f}{\partial t}(x,\xi_x)\right| \le M.$$

between t_n, t_0

BCT $\Rightarrow \lim_{n\to\infty}\int_a^b \underbrace{\frac{1}{t_n-t_0}(f(x,t_n)-f(x,t_0))}_{F_n(x)}\,dx$

$$= \int_a^b \lim_{n\to\infty}\frac{1}{t_n-t_0}(f(x,t_n)-f(x,t_0))\,dx$$

$$= \int_a^b \frac{\partial f}{\partial t}(x,t)\,dx.$$

- $\begin{cases} f_n \ge 0 \text{ measurable on } X \\ f_n \to f \text{ a.e. on } X \\ \int_X f_n\,d\mu \to \int_X f\,d\mu \text{ as } n\to\infty \end{cases}$ $\quad (X, \Sigma, \mu)$

$$\Rightarrow \forall E\in\Sigma,\quad \underbrace{\int_E f_n\,d\mu}_{\int_X f_n\chi_E} \to \int_E f\,d\mu.$$

Proof:

$$\int_E f\,d\mu \le \underline{\lim}\int_E f_n\,d\mu \quad \leftarrow (\text{Fatou})$$

$$\int_{X-E} f\,d\mu \le \underline{\lim}\int_{X-E} f_n\,d\mu$$

$$\cancel{\int_X f\,d\mu} - \int_E f\,d\mu \le \underline{\lim}\left(\int_X f_n\,d\mu - \int_E f_n\,d\mu\right)$$

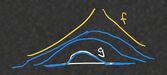
$$= \lim\int_X f_n\,d\mu - \overline{\lim}\int_E f_n\,d\mu$$

$$= \cancel{\int_X f\,d\mu} - \overline{\lim}\int_E f_n\,d\mu \quad (<\infty)$$

$$\overline{\lim}\int_E f_n\,d\mu \le \int_E f\,d\mu.$$

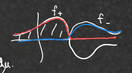
- $f \ge 0$

$$\int_X f\,d\mu := \sup_{g\le f}\int_X g\,d\mu$$



$S = \{g : \mu(\operatorname{supp}(g)) < \infty$ and g bounded and $0\le g\le f\}$

- $f = f_+ - f_-$



$$\int_X f\,d\mu = \int_X f_+\,d\mu - \int_X f_-\,d\mu.$$

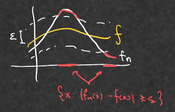
Convergence in measure

$\{f_n\}$ converges in measure to f

$\overset{\text{def}}{\Leftrightarrow}$ $\forall\varepsilon>0$, $\quad \lim_{n\to\infty}\mu\{x\in X : |f_n(x)-f(x)|\ge\varepsilon\} = 0$

$f_n \xrightarrow{L^p} f \quad \boxed{p\ge 1}$

$\overset{\text{def}}{\Leftrightarrow} \lim_{n\to\infty}\underbrace{\int_X |f_n - f|^p\,d\mu}_{\|f_n-f\|_p^p} = 0$



$\{x : |f_n(x)-f(x)|\ge\varepsilon\}$

L^p converges $\Rightarrow$ converges in measure.

$$\underbrace{\int_X |f_n - f|^p\,d\mu}_{\xrightarrow{n\to\infty} 0} \ge \int_{\{x\in X : |f_n(x)-f(x)|\ge\varepsilon\}} \overbrace{|f_n - f|^p}^{\ge\varepsilon^p}\,d\mu$$

$$\ge \varepsilon^p\,\underbrace{\mu\{x : |f_n(x)-f(x)|\ge\varepsilon\}}_{\searrow 0}.$$

Uniform convergence on X $\Rightarrow$ convergence in measure.

if $f_n \rightrightarrows f$ on X

Prop: $\mu(X) < +\infty$ $\quad \{f_n\}$ measurable

If $f_n \to f$ a.e. on X, then $f_n \xrightarrow{\text{meas}} f$.

If $f_n \xrightarrow{\text{meas}} f$, then $\exists\{f_{n_k}\}$ s.t. $f_{n_k} \to f$ a.e. on X.

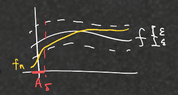
Proof: Egorov's Theorem

$\Rightarrow \forall\delta>0$, $\exists A_\delta\in\Sigma$ s.t. $\mu(A_\delta)<\delta$ and $f_n \rightrightarrows f$ on $X - A_\delta$

$$\mu(\underbrace{\{x : |f_n(x)-f(x)|\ge\varepsilon\}}_{\subset A_\delta})$$

$\le \mu(A_\delta)$ (for n large)

$< \delta$



$$\overline{\lim_{n\to\infty}}\,\mu(\{x : |f_n(x)-f(x)|\ge\varepsilon\}) \le \delta. \quad \forall\delta$$

$$\Rightarrow \overline{\lim_{n\to\infty}}\,\underbrace{\mu\{x : |f_n(x)-f(x)|\ge\varepsilon\}}_{\ge 0} = 0$$

$$\Rightarrow \lim_{n\to\infty}\mu\ \text{—}\ \text{—}\ = 0.$$

Converse: Assume $\lim_{n\to\infty}\mu\{x : |f_n(x)-f(x)|\ge\varepsilon\} = 0$. $\quad$ any.

Take $\varepsilon_k > 0$: $\quad \mu\{x : |f_n(x)-f(x)|\ge\varepsilon_k\} \xrightarrow{n\to\infty} 0$.

To be chosen. $\rightarrow \delta_k > 0$ $\quad \exists n_k$ s.t.

$$\mu\underbrace{\{x : |f_{n_k}(x)-f(x)|\ge\varepsilon_k\}}_{=:E_k} \le \delta_k.$$

$x\in\infty$-many E_k's $\Rightarrow \exists k_i\nearrow$ s.t. $\mu(E_{k_i}) \le \delta_{k_i}$.

$x\in$ finitely many E_k's $\Rightarrow$ for k large: $x\notin E_k$

$\Rightarrow |f_{n_k}(x) - f(x)| < \varepsilon_k$ for k large.

$$\therefore\ x\in X - \bigcap_{j=1}^{\infty}\bigcup_{k\ge j} E_k \Rightarrow \lim_{k\to\infty} f_{n_k}(x) = f(x).$$

$\mu = 0$?

need: $\sum_k \mu(E_k) < +\infty$.

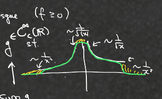
Choose δ_k s.t. $\sum_k \delta_k < \infty \Rightarrow \sum_k \mu(E_k) \le \sum_k \delta_k < +\infty$.

$(\mathbb{R}, \mu)$ $\quad f$ Lebesgue $(f\ge 0)$

Lebesgue

Thm: $\forall\varepsilon>0$, $\exists$ smooth $g\in C_c^\infty(\mathbb{R})$ s.t.

$$\underbrace{\int_{\mathbb{R}} |f-g|\,d\mu}_{\|f-g\|_{L^1}} < \varepsilon$$

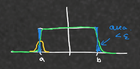


Supp g is compact

Proof:

Case ①: $\quad f = \chi_{(a,b)}$

e^{-x^2}



Case ②: $\quad f = \chi_A$, A measurable

A measurable $\Rightarrow \forall\varepsilon>0$, $\exists\, U = \bigcup(a_i, b_i) \supset A$ s.t. $\mu(U-A) < \varepsilon$.

$$\int_X |\chi_A - \chi_U| = \underbrace{\mu(A-U) + \mu(U-A)}_{<\varepsilon}$$

$$\int_X |\chi_U - \chi_{\bigcup_{i=1}^{N}(a_i,b_i)}| = \mu\left(\bigcup_{i>N}(a_i,b_i)\right) \to 0 \text{ as } N\to\infty.$$

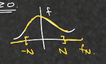
$\forall i = 1,\dots,N$, $\exists g_i\in C_c^\infty(\mathbb{R})$ $\quad < \varepsilon$ for N large.

$$\int_X |\chi_{(a_i,b_i)} - g_i| < \varepsilon$$

$$\Rightarrow \int_X \left|\chi_{\bigcup_{i=1}^{N}(a_i,b_i)} - \sum_{i=1}^{N} g_i\right| < \varepsilon N.$$

Case ③: $\quad f$ Lebesgue integrable ≥ 0.

$$f = \lim_{N\to\infty} f\chi_{[-N,N]}$$



$$f\chi_{[-N,N]} = \lim_{n\to\infty}\varphi_n^{(N)}$$

simple

$\forall k$, $\exists N_k$ s.t. $\left|f - f\chi_{[-N_k,N_k]}\right| < \frac{1}{k}$.

$\exists n_k$ s.t. $\left|f\chi_{[-N_k,N_k]} - \varphi_{n_k}^{(N_k)}\right| < \frac{1}{k}$.

$$\Rightarrow \left|f - \varphi_{n_k}^{(N_k)}\right| < \frac{2}{k} \to 0 \text{ as } k\to\infty.$$

$\therefore\ \varphi_{n_k}^{(N_k)} \to f$.

$$\lim_{k\to\infty}\int_X |f - \varphi_{n_k}^{(N_k)}| = 0.$$

$\forall\varepsilon$, $\exists k$ $\quad \left|\int_X f - \varphi_{n_k}^{(N_k)}\right| < \varepsilon$.

$\forall i$ $\quad \exists g_i$ s.t. $\int_X |\chi_{E_i} - g_i| < \varepsilon$

$$|f - \varphi_{n_k}^{(N_k)}| \le |f| + |\varphi_{n_k}^{(N_k)}| \le |f| + \left(\sum_i c_i\chi_{E_i}\right) \le |f| + \left(|f| + \frac{\varepsilon}{k}\right) \le |f|\left(|f| + 2\right).$$

$$\Rightarrow \int_X \left|\sum_i c_i\chi_{E_i} - \sum_i c_i g_i\right| < \sum_{i=1}^{N} c_i\varepsilon.$$