香港科技大學
THE HONG KONG
UNIVERSITY OF SCIENCE
AND TECHNOLOGY

數學系
DEPARTMENT OF MATHEMATICS

# MIDTERM EXAMINATION

**Course Code:** MATH 3043
**Course Title:** Honors Real Analysis
**Semester:** Spring 2018-19
**Date and Time:** 2:00PM-5:00PM, 26 October 2019

## Instructions

- Do **NOT** open the exam until instructed to do so.
- All mobile phones and communication devices should be switched **OFF**.
- It is an **OPEN-NOTES** exam. Only authorized materials specified in the midterm announcement are allowed.
- Answer **ALL** problems. Write your solutions in the space provided.
- You must **SHOW YOUR WORK** and **JUSTIFY YOUR STEPS** to receive credits in every problem in Part B.
- Some problems in Part B are structured into several parts. You can quote the results stated in the preceding parts to do the next part.

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## HKUST Academic Honor Code

Honesty and integrity are central to the academic work of HKUST. Students of the University must observe and uphold the highest standards of academic integrity and honesty in all the work they do throughout their program of study. As members of the University community, students have the responsibility to help maintain the academic reputation of HKUST in its academic endeavors. Sanctions will be imposed on students, if they are found to have violated the regulations governing academic integrity and honesty.

"I confirm that I have answered the questions using only materials specified approved for use in this examination, that all the answers are my own work, and that I have not received any assistance during the examination."

**Student's Signature:** Solution

**Student's Name:** ______________________

FAMILY NAME, First Name

**HKUST ID:** ______________ **Seat Number:** ______________

## Part A - Short Questions (20 points)

[Recommended time: $<$ 30 min.]

1. Who is the doctoral adviser of Henri Lebesgue? Put ✓ in the correct answer: **[1]**
   - ◯ Elias Stein
   - ☑ Émile Borel
   - ◯ Constantin Caratheodory
   - ◯ Augustin-Louis Cauchy
   - ◯ None of the above. Please write down: ____________________

2. Consider a countable sequence of subsets $\{E_{i,j}\}$ of $\mathbb{R}$ indexed by pairs $(i,j) \in \mathbb{N} \times \mathbb{N}$.

   (a) Which ONE of the following sets below is equal to the set $S_1$? **[2]**

   $$S_1 := \{x : \exists k \in \mathbb{N} \text{ such that whenever } i \geq k \text{ we have } x \in E_{i,j} \text{ for infinitely many } j\text{'s}\}$$

   Put ✓ in the correct answer:
   - ◯ $\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} \bigcap_{l=1}^{\infty} \bigcup_{j=l}^{\infty} E_{i,j}$
   - ◯ $\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} \bigcup_{l=1}^{\infty} \bigcap_{j=l}^{\infty} E_{i,j}$
   - ☑ $\bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} \bigcap_{l=1}^{\infty} \bigcup_{j=l}^{\infty} E_{i,j}$
   - ◯ $\bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} \bigcup_{l=1}^{\infty} \bigcap_{j=l}^{\infty} E_{i,j}$

   (b) Express the set $S_2$ below using: countable unions, countable intersections, complements (i.e. minus), and the sets $E_{i,j}$ and $\mathbb{R}$. **[3]**

   $$S_2 = \{x : \text{there exists infinitely many } i\text{'s such that } x \in E_{i,j} \text{ for finitely many } j\text{'s}\}$$

   $\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} \left(\mathbb{R} - \bigcap_{l=1}^{\infty} \bigcup_{j=l}^{\infty} E_{i,j}\right)$ or $\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} \bigcup_{l=1}^{\infty} \bigcap_{j=l}^{\infty} (\mathbb{R} - E_{i,j})$

3. Consider the following functions:

   (a) $f : \mathbb{R} \to \mathbb{R}$, $f = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \chi_{[n-1,n)}$. Put ✓ in ALL correct description(s): **[3]**
   - ☑ $f$ is a measurable function (with respect to the Lebesgue measure)
   - ☑ $f$ is (improper) Riemann integrable on $\mathbb{R}$.
   - ☑ $f$ is Lebesgue integrable on $\mathbb{R}$.

   (b) $g : \mathbb{R} \to \mathbb{R}$, $g = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \chi_{[n-1,n)}$. Put ✓ in ALL correct description(s): **[3]**
   - ☑ $g$ is a measurable function (with respect to the Lebesgue measure)
   - ☑ $g$ is (improper) Riemann integrable on $\mathbb{R}$.
   - ◯ $g$ is Lebesgue integrable on $\mathbb{R}$.

4. Give a **short** proof of each statement below.

(a) Show that any outer measure $\mu^*$ on a non-empty set $X$ is a complete measure. **[2]**

Suppose $S \subset X$ s.t. $\mu^*(S) = 0$,
then $\forall Y \subset X$, we have

$$\mu^*(Y \cap S) + \mu^*(Y - S) \leq \underbrace{\mu^*(S)}_{=0} + \mu^*(\underbrace{Y-S}_{\subset Y}) \leq \mu^*(Y).$$

$\therefore$ $S$ is measurable $\forall S$ with $\mu^*(S) = 0$. $\therefore$ $\mu$ is complete.

(b) Suppose $g : \mathbb{R} \to \mathbb{R}$ is a measurable function, show that $f(x) := g([x]) : \mathbb{R} \to \mathbb{R}$ is also measurable. Here $[x]$ denotes the largest integer less than or equal to $x$, e.g. $[\pi] = 3$, $[1.99] = 1$, $[2] = 2$. **[2]**

$f(x) = g(n)$ if $x \in [n, n+1)$, $n \in \mathbb{Z}$.

$\therefore$ $f(x) = \sum_{n \in \mathbb{Z}} g(n) \chi_{\underbrace{[n,n+1)}_{\text{measurable set}}}$ $\Rightarrow$ $\chi_{[n,n+1)}$ is measurable

$\Rightarrow$ $f(x) = \lim_{N \to \infty} \sum_{n=0}^{N} g(n) \chi_{[n,n+1)} + \lim_{N \to -\infty} \sum_{-N}^{-1} g(n) \chi_{[n,n+1)}$ is measurable.

5. Show that **[4]**

$$f(t) := \int_{-\infty}^{\infty} e^{-x^2} \sin(tx^2)\, dx$$

is a continuous function of $t$.

Consider any sequence $t_n \to t_0 \in \mathbb{R}$.

let $F_n(x) = e^{-x^2} \sin(t_n x)$, then

$|F_n(x)| \leq e^{-x^2} =: g(x)$

$\int_{-\infty}^{\infty} e^{-x^2} dx < \infty$ $\Rightarrow$ By LDCT, $\lim_{n \to \infty} \int_{\mathbb{R}} F_n(x) dx$

$= \int_{\mathbb{R}} \lim_{n \to \infty} e^{-x^2} \sin(t_n x^2) dx$

$= \int_{\mathbb{R}} e^{-x^2} \underbrace{\sin(t_0 x^2)}_{\text{cts.}} dx$

$\therefore$ $\lim_{n \to \infty} f(t_n) = f(t_0)$

$\forall$ sequence $t_n \to t_0$.

$\Rightarrow$ $f(t)$ is continuous at any $t = t_0$.

## Part B - Long Questions (80 points): Answer ALL FOUR problems

[Recommended time: Q1 $<$ 30 min, Q2 $<$ 30 min, Q3 $<$ 45 min, Q4 $<$ 45 min]

1. Given a $C^1$ vector field $F(x,y) = (f_1(x,y), f_2(x,y)) : \mathbb{R}^2 \to \mathbb{R}^2$ such that its Jacobian matrix at $(0,0)$ is given by: **[~~20~~ 15]**

$$DF(0,0) = P^T \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} P$$

where $P$ is a fixed $2 \times 2$ invertible matrix of real entries. Suppose further that $F(0,0) = (0,0)$. Let $(x(t), y(t))$ be the solution to the ODE system

$$x'(t) = f_1(x(t), y(t))$$
$$y'(t) = f_2(x(t), y(t))$$

Show that there exists $\varepsilon > 0$ such that whenever $0 < \sqrt{x(t)^2 + y(t)^2} < \varepsilon$ at $t$, we have

$$\frac{d}{dt}\left(x(t)^2 + y(t)^2\right) < 0 \text{ at } t.$$

2. Consider a $C^1$ function $f(x,y,z) : \mathbb{R}^3 \to \mathbb{R}$ such that $\Sigma := f^{-1}(0)$ is non-empty and $\nabla f(p)$ is non-zero for any $p \in \Sigma$. Show that for any $p \in \Sigma$, there exists a **bijective** map $\psi : U \to V$ from an open set $U \subset \mathbb{R}^3$ containing $p$ to another open set $V \subset \mathbb{R}^3$ so that **both** $\psi$ and its inverse $\psi^{-1}$ are $C^1$, and that **[10]**

$$\Sigma \cap U = \{\psi^{-1}(x,y,0) : (x,y,0) \in V\}.$$

[Hint: Draw a picture first!]

3. Consider a sequence of measurable functions $f_n : \mathbb{R} \to \mathbb{R}$ with respect to the Lebesgue measure $\mu$ on $\mathbb{R}$. Given that for each $n, k \in \mathbb{N}$, we have

$$\mu\left(\left\{x \in \mathbb{R} : |f_n(x)| > \frac{1}{k}\right\}\right) \leq \frac{1}{2^n}.$$

Consider the sets

$$E_k^+ := \left\{x \in \mathbb{R} : \limsup_{n\to\infty} f_n(x) > \frac{1}{k}\right\}$$
$$E_k^- := \left\{x \in \mathbb{R} : \liminf_{n\to\infty} f_n(x) < -\frac{1}{k}\right\}$$

   (a) Suppose $x \in \mathbb{R} - \bigcup_{k=1}^{\infty}(E_k^+ \cup E_k^-)$. Show that $f_n(x) \to 0$ as $n \to \infty$. **[5]**

   (b) Show that $f_n \to 0$ a.e. on $\mathbb{R}$. **[20]**

4. (a) Denote by $\mathcal{H}^n(A)$ the $n$-dimensional Hausdroff measure of a set $A \subset \mathbb{R}^n$, where $n \in \mathbb{N}$. Show that $\mathcal{H}^n([-N,N]^n) < +\infty$ for any square cube $[-N,N]^n \subset \mathbb{R}^n$ where $N \in \mathbb{N}$. **[10]**

   (b) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on a measure space $(\mathbb{R}^n, \Sigma, \mathcal{H}^n)$, where $\Sigma$ is the $\sigma$-algebra of all $\mathcal{H}^n$-measurable sets in $\mathbb{R}^n$. Suppose $f_n \to f$ a.e. on $\mathbb{R}^n$ to a measurable function $f$. Show that there exist countably many sets $E_k \in \Sigma$ and a set $S \in \Sigma$ with $\mu(S) = 0$, such that **[~~25~~ 20]**

$$\mathbb{R}^n \; \cancel{X} = S \cup \left(\bigcup_{k=1}^{\infty} E_k\right)$$

and $f_n$ converges to $f$ uniformly on each $E_k$.

* End of Paper *

1. Given a $C^1$ vector field $F(x,y) = (f_1(x,y), f_2(x,y)) : \mathbb{R}^2 \to \mathbb{R}^2$ such that its Jacobian matrix at $(0,0)$ is given by: [~~10~~] [15]

$$DF(0,0) = P^T \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} P$$

where $P$ is a fixed $2 \times 2$ invertible matrix of real entries. Suppose further that $F(0,0) = (0,0)$. Let $(x(t), y(t))$ be the solution to the ODE system

$$x'(t) = f_1(x(t), y(t))$$
$$y'(t) = f_2(x(t), y(t))$$

Show that there exists $\varepsilon > 0$ such that whenever $0 < \sqrt{x(t)^2 + y(t)^2} < \varepsilon$ at $t$, we have

$$\frac{d}{dt}\left(x(t)^2 + y(t)^2\right) < 0 \text{ at } t.$$

$f_1, f_2$ are $C^1$ $\Rightarrow$ $\begin{cases} f_1(x,y) = \cancel{f_1(0,0)}^{\,0} + \frac{\partial f_1}{\partial x}(0,0)\cdot x + \frac{\partial f_1}{\partial y}(0,0)\cdot y + E_1(x,y) \\ f_2(x,y) = \cancel{f_2(0,0)}^{\,0} + \frac{\partial f_2}{\partial x}(0,0)x + \frac{\partial f_2}{\partial y}(0,0)\cdot y + E_2(x,y) \end{cases}$

where $|E_i(x,y)| = o(\sqrt{x^2+y^2})$, $i=1,2$.

$$\frac{d}{dt}(x(t)^2 + y(t)^2) = 2x(t)x'(t) + 2y(t)y'(t) = 2\,[x(t)\ \ y(t)]\begin{bmatrix} x'(t) \\ y'(t)\end{bmatrix}$$

$$= 2\,[x(t)\ \ y(t)]\,\vec{F}(x(t), y(t))$$

$$= 2\,[x(t)\ \ y(t)]\left(\cancel{F(0,0)}^{\,0} + DF(0,0)\begin{bmatrix} x(t) \\ y(t)\end{bmatrix} + o\left(\sqrt{x(t)^2+y(t)^2}\right)\right)$$

$$\leq 2\,[x(t)\ \ y(t)]\,DF(0,0)\begin{bmatrix} x(t) \\ y(t)\end{bmatrix} + o\left(x(t)^2+y(t)^2\right)$$

$\|[x(t), y(t)]\| = \sqrt{x(t)^2+y(t)^2}$

$$= 2\,[x(t)\ \ y(t)]\,P^T\begin{bmatrix} -2 & 0 \\ 0 & -3\end{bmatrix}P\begin{bmatrix} x(t) \\ y(t)\end{bmatrix} + o\left(x(t)^2+y(t)^2\right)$$

$$= 2\left(P\begin{bmatrix} x(t) \\ y(t)\end{bmatrix}\right)^T\begin{bmatrix} -2 & 0 \\ 0 & -3\end{bmatrix}P\begin{bmatrix} x(t) \\ y(t)\end{bmatrix} + o\left(x(t)^2+y(t)^2\right)$$

let $\begin{bmatrix} \tilde{x}(t) \\ \tilde{y}(t)\end{bmatrix} = P\begin{bmatrix} x(t) \\ y(t)\end{bmatrix}$, then

$$\frac{d}{dt}(x(t)^2+y(t)^2) \leq 2\,[\tilde{x}(t), \tilde{y}(t)]\begin{bmatrix} -2 & 0 \\ 0 & -3\end{bmatrix}\begin{bmatrix} \tilde{x}(t) \\ \tilde{y}(t)\end{bmatrix} + E(x(t), y(t))$$

$$= -4\tilde{x}(t)^2 - 6\tilde{y}(t)^2 + E(x(t), y(t))$$

$P^{-1}\begin{bmatrix}\tilde{x} \\ \tilde{y}\end{bmatrix} = \begin{bmatrix} x \\ y\end{bmatrix}$

$\Rightarrow x^2+y^2 \leq \|P^{-1}\|^2(\tilde{x}^2+\tilde{y}^2)$

$$\leq -6\left(\tilde{x}(t)^2 + \tilde{y}(t)^2\right) + E(x(t), y(t))$$

$$\leq -\frac{6}{\|P^{-1}\|^2}\left(x(t)^2+y(t)^2\right) + o\left(x(t)^2+y(t)^2\right) < 0$$

when $x(t)^2+y(t)^2$ small. $\boxed{\text{Q}}$

2. Consider a $C^1$ function $f(x,y,z):\mathbb{R}^3\to\mathbb{R}$ such that $\Sigma := f^{-1}(0)$ is non-empty and $\nabla f(p)$ is non-zero for any $p\in\Sigma$. Show that for any $p\in\Sigma$, there exists a **bijective** map $\psi: U\to V$ from an open set $U\subset\mathbb{R}^3$ containing $p$ to another open set $V\subset\mathbb{R}^3$ so that **both** $\psi$ and its inverse $\psi^{-1}$ are $C^1$, and that **[10]**

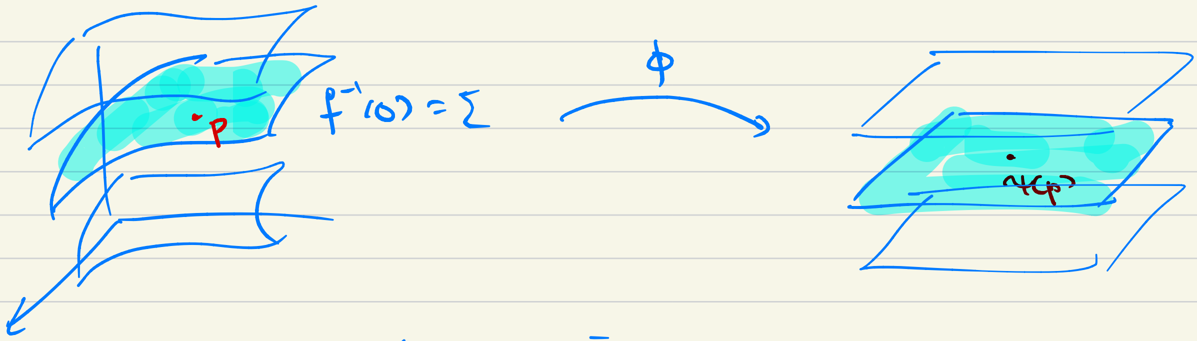
$$\Sigma\cap U = \{\psi^{-1}(x,y,0) : (x,y,0)\in V\}.$$

[Hint: Draw a picture first!]

At $p$, $\nabla f(p)\neq\vec{0} \implies$ WLOG assume $\frac{\partial f}{\partial z}(p)\neq 0$.

Define a "straightening" map $\phi:\mathbb{R}^3\to\mathbb{R}^3$ by:

$$\phi(x,y,z) = (x,y,f(x,y,z))$$




then $$D\phi = \frac{\partial(x,y,f)}{\partial(x,y,z)} = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\end{bmatrix}$$

$\therefore \det D\phi(p) = \frac{\partial f}{\partial z}(p)\neq 0.$

By inverse function theorem, $\exists\, \mathcal{U}\ni p$ and $V\ni\phi(p)$ s.t.

$\psi := \phi|_{\mathcal{U}} : \mathcal{U}\to V$ is a diffeomorphism.

Claim: $\Sigma\cap\mathcal{U} = \{\psi^{-1}(x,y,0) : (x,y,0)\in V\}$.

Proof: $\underset{(x,y,z)}{q} \in \underset{f^{-1}(0)}{\Sigma}\cap\mathcal{U} \Rightarrow f(q)=0 \Rightarrow \psi(q) = (x,y,f(\underbrace{x,y,z}_{q})) = \underbrace{(x,y,0)}_{\in V}$

$\therefore q\in\{\psi^{-1}(x,y,0): (x,y,0)\in V\}$

$\Rightarrow \Sigma\cap\mathcal{U}\subset\{\psi^{-1}(x,y,0): (x,y,0)\in V\}$

Conversely, if $\underset{(x,y,z)}{q} = \psi^{-1}(x,y,0)\overset{\in\mathcal{U}}{=}\exists (x,y,0)\in V$, then

$\psi(q) = (x,y,0) \Rightarrow (x,y,f(\underbrace{x,y,z}_{q})) = (x,y,z) \Rightarrow f(q)=0 \Rightarrow q\in\Sigma.$

$\therefore q\in\Sigma\cap\mathcal{U}. \Rightarrow \Sigma\cap\mathcal{U} = \{\psi^{-1}(x,y,0) : (x,y,0)\in V\}.$

3. Consider a sequence of measurable functions $f_n : \mathbb{R} \to \mathbb{R}$ with respect to the Lebesgue measure $\mu$ on $\mathbb{R}$. Given that for each $n, k \in \mathbb{N}$, we have

$$\mu\left(\left\{x \in \mathbb{R} : |f_n(x)| > \frac{1}{k}\right\}\right) \leq \frac{1}{2^n}.$$

Consider the sets

$$E_k^+ := \left\{x \in \mathbb{R} : \limsup_{n\to\infty} f_n(x) > \frac{1}{k}\right\}$$

$$E_k^- := \left\{x \in \mathbb{R} : \liminf_{n\to\infty} f_n(x) < -\frac{1}{k}\right\}$$

(a) Suppose $x \in \mathbb{R} - \bigcup_{k=1}^{\infty}(E_k^+ \cup E_k^-)$. Show that $f_n(x) \to 0$ as $n \to \infty$. **[5]**

(b) Show that $f_n \to 0$ a.e. on $\mathbb{R}$. **[20]**

(a) Suppose $x \in \mathbb{R} - \bigcup_{k=1}^{\infty}(E_k^+ \cup E_k^-) = \bigcap_{k=1}^{\infty}\left((\mathbb{R}-E_k^+) \cap (\mathbb{R}-E_k^-)\right)$

then $\forall k \in \mathbb{N}$, $x \in \mathbb{R} - E_k^+$ and $x \in \mathbb{R} - E_k^-$

$$\Rightarrow \underbrace{-\frac{1}{k} \leq \varliminf_{n\to\infty} f_n(x)}_{x \notin E_k^-} \leq \underbrace{\varlimsup_{n\to\infty} f_n(x) \leq \frac{1}{k}}_{x \notin E_k^+} \quad \forall k \in \mathbb{N}.$$

let $k \to \infty$, then $0 \leq \varliminf_{n\to\infty} f_n(x) \leq \varlimsup_{n\to\infty} f_n(x) \leq 0$

$\therefore \lim_{n\to\infty} f_n(x) = 0$.

(b) We argue that $\mu\left(\bigcup_{k=1}^{\infty}(E_k^+ \cup E_k^-)\right) = 0$:

When $x \in E_k^+$, we have

$\varlimsup_{n\to\infty} f_n(x) > \frac{1}{k}$

$\frac{1}{k}$ $\overline{\lim}$

From 2043, $\exists$ infinitely many $n$'s

s.t. $f_n(x) > \frac{1}{k} \Rightarrow |f_n(x)| > \frac{1}{k}$

Denote $F_{n,k} := \{x \in \mathbb{R} : |f_n(x)| > \frac{1}{k}\}$

then $x \in E_k^+ \Rightarrow x \in F_{n,k}$ for infinitely many $n$'s.

$$\therefore \boxed{E_k^+ \subset \bigcap_{m=1}^{\infty} \bigcup_{n>m} F_{n,k}}$$

Similarly, if $x \in E_k^-$, then $\varliminf_{n\to\infty} f_n(x) < -\frac{1}{k} \Rightarrow \exists$ infinitely many $n$'s s.t. $f_n(x) < -\frac{1}{k}$ $\Rightarrow |f_n(x)| > \frac{1}{k}$.

$$\boxed{\therefore E_k^- \subset \bigcap_{m=1}^{\infty} \bigcup_{n>m} F_n}$$

$$\sum_{n=1}^{\infty} \mu(F_{n,k}) \underset{\substack{\uparrow \\ \text{given}}}{\leq} \sum_{n=1}^{\infty} \frac{1}{2^n} = 2 < \infty$$

$\xRightarrow{\text{Borel-Cantelli}}$

$$\mu\left(\bigcap_{m=1}^{\infty} \bigcup_{n \geq m} F_{n,k}\right) = 0 . \quad \forall k$$

$$\Rightarrow \quad \mu(E_k^+ \cup E_k^-) \leq \mu\left(\bigcap_{m=1}^{\infty} \bigcup_{n \geq m} F_{n,k}\right) = 0 \quad \forall k$$

$$\Rightarrow \quad \mu\left(\bigcup_{k=1}^{\infty} (E_k^+ \cup E_k^-)\right) \leq \sum_{k=1}^{\infty} \underbrace{\mu(E_k^+ \cup E_k^-)}_{=0} = 0 .$$

$$f_n(x) \longrightarrow 0 \quad \forall x \notin \underbrace{\bigcup_{k=1}^{\infty} (E_k^+ \cup E_k^-)}_{\mu = 0} , \Rightarrow f_n \to 0 \text{ a.e. on } \mathbb{R} .$$

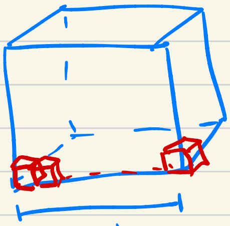
4. (a) Denote by $\mathcal{H}^n(A)$ the $n$-dimensional Hausdroff measure of a set $A \subset \mathbb{R}^n$, where $n \in \mathbb{N}$. Show that $\mathcal{H}^n([-N,N]^n) < +\infty$ for any square cube $[-N,N]^n \subset \mathbb{R}^n$ where $N \in \mathbb{N}$. **[10]**

(b) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on a measure space $(\mathbb{R}^n, \Sigma, \mathcal{H}^n)$, where $\Sigma$ is the $\sigma$-algebra of all $\mathcal{H}^n$-measurable sets in $\mathbb{R}^n$. Suppose $f_m \to f$ a.e. on $\mathbb{R}^n$ to a measurable function $f$. Show that there exist countably many sets $E_k \in \Sigma$ and a set $S \in \Sigma$ with $\cancel{\mu}\ \mathcal{H}^n(S) = 0$, such that **[~~25~~ 20]**

$$\mathbb{R}^n\ \cancel{X} = S \cup \left(\bigcup_{k=1}^{\infty} E_k\right)$$

and $f_n$ converges to $f$ uniformly on each $E_k$.

(a)



$= [-N,N]^n$

$\underbrace{\qquad}_{2N}$

For any $\delta > 0$, we consider closed cubes $\underset{C}{\boxed{\ }}$ with length $\frac{\delta}{2\sqrt{n}}$ s.t.

$\text{diam}(C) = \frac{\delta}{2} < \delta$.

We can cover $[-N,N]^n$ by $\left\lceil \frac{2N}{\left(\frac{\delta}{2\sqrt{n}}\right)} \right\rceil^n$ $\left\lceil \frac{2N}{\delta/2\sqrt{n}} \right\rceil$ many cubes with $\text{diam}(C) = \frac{\delta}{2}$,

then $H^n_\delta([-N,N]^n) = \inf\left\{ \sum_j \left(\frac{\text{diam}\, C_j}{2}\right)^n : \begin{array}{l}\text{diam}\, C_j < \delta \\ [-N,N]^n \subset \bigcup_j C_j\end{array} \right\}$

$$\le \underbrace{\left\lceil \frac{2N}{\frac{\delta}{2\sqrt{n}}} \right\rceil^n}_{\#\text{ of cubes}} \cdot \underbrace{\left(\frac{\delta/2}{2}\right)^n}_{=\left(\frac{\text{diam cube}}{2}\right)^n} \le \left(\frac{2N}{\frac{\delta}{2\sqrt{n}}} + 1\right)^n \left(\frac{\delta}{4}\right)^n = \left(\sqrt{n}N + \frac{\delta}{4}\right)^n$$

$$H^n([-N,N]^n) = \lim_{\delta \to 0^+} H^n_\delta([-N,N]^n) \le \lim_{\delta \to 0^+} \left(\sqrt{n}N + \frac{\delta}{4}\right)^n = (\sqrt{n}N)^n.$$

$< +\infty$.

(b) Consider each cube $[-N,N]^n$ which has finite $H^n$.

As $[-N,N]^n$ is a Borel set, it is $H^n$-measurable. By (a), $H^n([-N,N]^n) < +\infty$. By Egorff's Theorem (applied on $[-N,N]^n$):

For each $k \in \mathbb{N}$, $\exists A_{N,k} \subset [-N,N]^n$ with $H^n(A_{N,k}) < \frac{1}{k}$ $\varepsilon^2$

s.t. $f_m \rightrightarrows f$ on $E_{N,k} := [-N,N]^n - A_{N,k}$

Egorff's requires finite measure !!

$$\mathbb{R}^n = \bigcup_{N=1}^{\infty} [-N,N]^n = \bigcup_{N=1}^{\infty}\left(\left([-N,N]^n - \bigcap_{k=1}^{\infty} A_{N,k}\right) \cup \bigcap_{k=1}^{\infty} A_{N,k}\right)$$

$$= \bigcup_{N=1}^{\infty} \bigcup_{k=1}^{\infty} \underbrace{\left([-N,N]^n - A_{N,k}\right)}_{E_{N,k}} \cup \bigcup_{N=1}^{\infty}\bigcap_{k=1}^{\infty} A_{N,k}$$

Take $S := \underbrace{\bigcup_{N=1}^{\infty}\bigcap_{k=1}^{\infty}}_{\text{countable}} \underbrace{A_{N,k}}_{\in \Sigma} \in \Sigma$. We claim $\mathcal{H}^n(S) = 0$

$$\forall N \in \mathbb{N},\quad \mathcal{H}^n\left(\bigcap_{k=1}^{\infty} A_{N,k}\right) \underset{\forall k \in \mathbb{N}}{\leq} \mathcal{H}^n(A_{N,k}) \underset{\text{our choice}}{\leq} \frac{1}{k} \longrightarrow 0 \text{ as } k \to \infty.$$

$$\therefore\ \mathcal{H}^n\left(\bigcap_{k=1}^{\infty} A_{N,k}\right) = 0 \quad \forall N \in \mathbb{N}.$$

$$\Rightarrow \mathcal{H}^n\Big(\underbrace{\bigcup_{N=1}^{\infty}\bigcap_{k=1}^{\infty} A_{N,k}}_{S}\Big) \leq \sum_{N=1}^{\infty} \underbrace{\mathcal{H}^n\left(\bigcap_{k=1}^{\infty} A_{N,k}\right)}_{=0} = 0$$

$\therefore$ $\exists$ doubly indexed countably many $E_{N,k}$'s $\in \Sigma$, s.t.

$f_m \rightrightarrows f$ on each $E_{N,k}$, and

$$\mathbb{R}^n = \left(\bigcup_{\substack{N=1\\k=1}}^{\infty} E_{N,k}\right) \cup S$$

$\mathcal{H}^n(S) = 0$.

$S \in \Sigma$.