Tutorial 6 Problems

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Recollections.

Last week you learned the definition of Lebesgue integration. Recall the following key points:

- Any measurable function can be approximated by simple functions.
- Egorov's theorem. This is of fundamental importance in the theory of integrations, since other convergence theorems essentially come from this theorem.
- Definition of integration for simple functions: it can be proved that the obvious definition one can write down is well-defined.
- Definition of integration of nonnegative functions: taking sup among all the nonnegative simple functions less than the function.
- Definition of integration of general functions: separate the positive and negative parts.
- Properties of integration.
- Lebesgue Control Convergence theorem: $f_n \to f$ a.e., $|f_n| \leq g$, g integrable $\Rightarrow \int f_n \to \int f$. This is the most important theorem in the theory of Lebesgue integrations.

1 Warm Up.

1.1 ***A Dirty Trick in Linear Algebra.

G_{rss} is dense in G. One can use analysis to solve problems in linear algebra.

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1.2 Use Fatou's Lemma to prove Lebesgue Control Convergence theorem.

Recall Fatou's Lemma: $\int \liminf f_n \leq \liminf \int f_n$

1.3 Chebyshev's Inequality.

$f \geq 0$ integrable, $a > 0$, $E_a = \{x : f(x) > a\}$. Then $\mu(E_a) \leq \frac{1}{a} \int f$. This very simple result has some implication in probability theory: it directly implies the **Law of large numbers**.

1.4 Deal with Sets by Integrations.

$E_1, \cdots, E_n, E$ measurable sets, $\mu(E) < \infty$. Every point in E belongs to at least k of the E_i's. Then there exists some i s.t. $\mu(E_i) \geq \frac{k}{m}\mu(E)$.

2 Proof of Egorov's Theorem.

$\mu(E) < \infty$. On E, measurable functions $f_n \to f$ a.e. $\forall \epsilon > 0$, $\exists A \subseteq E$ closed s.t. $f_n \to f$ uniformly on A and $\mu(E - A) < \epsilon$.

3 The Graph of a Measurable Function

$f : \mathbf{R} \to \mathbf{R}$ measurable. $\Gamma_f = \{(x, f(x)) \in \mathbf{R}^2 : x \in \mathbf{R}\}$. Then Γ_f is a measurable set in $\mathbf{R}^2$ with measure 0.

4 The Boss

$f \geq 0$ measurable. $F_k = \{x : 2^k < f(x) \leq 2^{k+1}\}$, $E_k = \{x : f(x) > 2^k\}$. Then f is integrable iff. $\sum_{k \in \mathbf{Z}} 2^k \mu(F_k) < \infty$ iff. $\sum_{k \in \mathbf{Z}} 2^k \mu(E_k) < \infty$.