Lebesgue Integrals d'après Stein.

0. Measurable Functions

We restrict attention on $\mathbb{R}$, though the theory works for $\mathbb{R}^n$, even a general measure space.

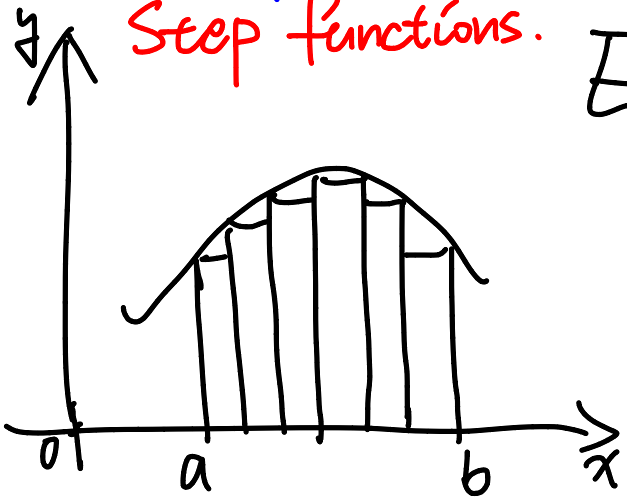
△ (Simple Functions) $E \subseteq \mathbb{R}$. $1_E : \mathbb{R} \to \{0,1\}$

$$1_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

simple function: $\sum_{k=1}^{N} a_i 1_{E_i}$. E_i measurable.

△ (Example) $a = x_0 < x_1 < \cdots < x_N = b$.

Step functions.



$E_k = [x_{k-1}, x_k)$ $\quad k = 1, \cdots, N$.

$a_k = f(x_{k-1})$

This is used in Riemann integral.

☆ (Measurable Func.) $E \subseteq \mathbb{R}$ measurable set.

$f : E \to \mathbb{R}$ measurable if $\underline{f^{-1}(-\infty, a)}$ is measurable for $\forall a \in \mathbb{R}$.
↳ denoted $\{f(x) < a\}$

△ (Example) Simple functions. Continuous functions.

△ (Prop) (i) $\{f_n\}$ measurable. then

$\sup_n f_n$, $\inf_n f_n$, $\overline{\lim_n} f_n$, $\underline{\lim_n} f_n$ measurable.

(ii) f, g measurable $\Rightarrow$ $f \pm g$, f^k, fg measurable.

Pf (i) Let $E_n = \{f_n > a\}$. then

$\{\sup_n f_n > a\} = \bigcup_n E_n$. similar for inf.

$\{\overline{\lim_n} f_n > a\} = \overline{\lim_n} E_n$. Similar for $\underline{\lim}$.

(ii) $\{f+g > a\} = \bigcup_{r \in \mathbb{Q}} (\{f > a-r\} \cap \{g > r\})$,

for f^k, divide into cases k odd/even.

$fg = \frac{1}{4}((f+g)^2 - (f-g)^2)$. $\square$

△ (Almost Everywhere) P_x: statement, $x \in \mathbb{R}$.

P_x holds a.e. $\overset{\text{def}}{\Leftrightarrow}$ $\mu(\{x \in \mathbb{R} : P_x \text{ is false}\}) = 0$.

Example: f measurable, $f = g$ a.e. $\Rightarrow$ g measurable

A lot of statements still hold in the sense of a.e.

☆(Key Approximation Result)

Measurable functions can be approximated by simple functions.

Namely, $f: \mathbb{R} \to \mathbb{R}_{\geq 0}$ measurable $\Rightarrow \exists \{\varphi_k\}_{k\geq 1}$ simple s.t. $\varphi_k \leq \varphi_{k+1}$ and $\varphi_k \to f$ pointwisely.

Pf. For $k \in \mathbb{Z}_{>0}$, $j = 0, 1, \cdots, k2^k - 1$, let

$$E_{k,j} = \{x \in \mathbb{R} : f(x) \in [\tfrac{j}{2^k}, \tfrac{j+1}{2^k})\}$$

$$E_{k,k2^k} = \{x \in \mathbb{R} : f(x) \geq k\}$$

These sets are mutually disjoint measurable, and $\bigcup_{j=0}^{k2^k} E_{k,j} = \mathbb{R}$. Define

$$\varphi_k(x) = \sum_{j=0}^{k2^k} \frac{j}{2^k} \varphi_{E_{k,j}}(x).$$

Exercise: Verify φ_k satisfies the properties we need. □

△(Cor) $f: \mathbb{R} \to \mathbb{R}$ measurable $\Rightarrow \exists \{\varphi_k\}_{k\geq 1}$ simple. s.t. $|\varphi_k| \leq |\varphi_{k+1}|$, $\varphi_k \to f$.

Pf $f^+ \overset{\text{def}}{=} \max\{f, 0\}$, $f^- \overset{\text{def}}{=} \max\{-f, 0\}$.

Then $f^+, f^- \geq 0$, $f = f^+ - f^-$ □

Mostly this trick reduces discussions of a general function to the case of nonnegative functions.

☆(Egorov) $E \subseteq \mathbb{R}$ measurable. $\mu(E) < +\infty$

$\{f_k\}_{k\geq 1}$ measurable functions on E. $f_k \xrightarrow{\text{ptwise}} f_{a.e.}$

$\Rightarrow \forall \varepsilon > 0$, $\exists A \subseteq E$ closed s.t. $\mu(E-A) < \varepsilon$

and $f_k \xrightarrow{\text{uniform}} f$ on A.

Pf $\{x \in E : f_k(x) \not\to f(x)\}$ has measure 0

$$\| \\ \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n\geq N} \{x \in E : |f_n(x) - f(x)| \geq \tfrac{1}{k}\}. \Rightarrow \forall k \geq 1,$$

$$\lim_{N\to\infty} \mu\Big(\bigcup_{n\geq N} \{x \in E : |f_n(x) - f(x)| \geq \tfrac{1}{k}\}\Big) = 0 \Rightarrow$$

$\exists N_k$ s.t. $\mu\Big(\bigcup_{n\geq N_k} \{x \in E : |f_n(x) - f(x)| \geq \frac{1}{k}\}\Big) < \frac{\varepsilon}{2^{k+1}}$.

Let $e_k = \bigcup_{n\geq N_k} \{x \in E : |f_n(x) - f(x)| \geq \frac{1}{k}\}$. $e = \bigcup_{k=1}^{\infty} e_k$

Then $\mu(e) \leq \frac{\varepsilon}{2}$, and $f_k \xrightarrow{\text{uniform}} f$ on $E-e$

(To see this. $\forall x \in E-e$, $x \notin e_k$ for $\forall k \Rightarrow$
when $n \geq N_k$, $|f_n(x) - f(x)| < \frac{1}{k}$ for all $x \in E-e \Rightarrow$
uniform convergence.)

Finally, take a closed set $A \subseteq E-e$ s.t.
$\mu(E-e-A) < \frac{\varepsilon}{2}$. $\square$

1. Integrations

partition

Your previous approach: $E = \bigsqcup_{i \in I} E_i$, $x_i^* \in E_i$

Consider the limit of $\sum_{i \in I} f(x_i^*)\mu(E_i)$

in the direct system of partitions.

$\leftarrow$ Direct generalization of Riemann integral.

Another approach: step by step.

Step 1. Simple functions.

$\triangle$ (Canonical Form) φ: simple function on $\mathbb{R}$

The way of writing $\varphi = \sum_{i=1}^{N} a_i 1_{E_i}$ is not unique.

It is called canonical if $E_1 \sim E_N$ mutually disjoint, $a_1 \sim a_N$ are distinct. There $\exists 1$ canonical form.

Pf Since φ is simple, $\mathrm{Im}\,\varphi$ is a finite set.

Let it be $\{c_1, \cdots, c_N\}$, let $E_i = \varphi^{-1}(c_i)$.

Then $\varphi = \sum_{i=1}^{N} c_i 1_{E_i}$ is canonical.

Uniqueness: If there are two canonical forms

$\sum_{i=1}^{M} a_i 1_{E_i} = \sum_{j=1}^{N} b_j 1_{F_j}$. $\{a_1, \cdots, a_M\} = \{b_1, \cdots, b_N\} = \mathrm{Im}\,\varphi$

$\Rightarrow$ We can write $\sum_{i=1}^{N} a_i 1_{E_i} = \sum_{i=1}^{N} a_i 1_{F_i}$

The rest is clear. $\square$

☆ $\varphi = \sum_{i=1}^{N} a_i 1_{E_i}$ canonical. define

$\int_{\mathbb{R}} \varphi\, dx = \sum_{i=1}^{N} a_i \mu(E_i)$. For $E \subseteq \mathbb{R}$ measurable

define $\int_E \varphi\, dx = \int_{\mathbb{R}} \underline{\varphi 1_E}\, dx$ still simple.

☆ (Properties) (i) For any representation

$\varphi = \sum_{i=1}^{N} c_i 1_{F_i}$ (not necessarily canonical)

we have $\int \varphi = \sum_{i=1}^{N} c_i \mu(F_i)$.

(ii) Linearity $\int (a\varphi + b\psi) = a\int\varphi + b\int\psi$.

(iii) Additivity $E \cap F = \phi \Rightarrow \int_{E\cup F} \varphi = \int_E \varphi + \int_F \varphi$.

(iv) Monotonicity $\varphi \le \psi \Rightarrow \int \varphi \le \int \psi$.

(v) Triangle Inequality $\left|\int \varphi\right| \le \int |\varphi|$.

(vi) $\varphi = 0$ a.e. $\Rightarrow \int \varphi = 0$.

Pf (i) For an arbitrary representation $\sum_{i=1}^{N} c_i 1_{F_i}$.

first write $\bigcup_{i=1}^{N} F_i$ into mutually disjoint parts

$\bigcup_{j=1}^{M} G_j$. then $\varphi = \sum_{j=1}^{N} b_j 1_{G_j}$, and $\sum_{j=1}^{M} b_j \mu(G_j) = \sum_{i=1}^{N} c_i \mu(F_i)$

Then we merge the terms $b_j 1_{G_j}$ with the same b_j. the sum is still not changed, and the resulting representation is canonical. $\square$

(ii) Use (i).

(iii) If $E \cap F = \phi$, then $1_{E \cup F} = 1_E + 1_F$. Use (ii).

(iv) $\psi - \varphi \geq 0 \Rightarrow$ in its canonical form $\sum c_i 1_{E_i}$, each $c_i \geq 0 \Rightarrow \int (\psi - \varphi) \geq 0$.

(v) Use (i)

(vi) In the canonical form $\sum c_i 1_{E_i}$, each E_i has measure 0. $\square$

Step 2. Bounded functions supported on a set of finite measure.

$\triangle$ f: measurable. $\operatorname{supp} f \overset{\text{def}}{=} \{x \in \mathbb{R} : f(x) \neq 0\}$.

f is **supported** on $E \overset{\text{def}}{\Longleftrightarrow} \operatorname{supp} f \subseteq E$.

We say f is **S2** if f is bounded and $\operatorname{supp} f$ has finite measure.

$\triangle$ (Lem) f is S2. $\operatorname{supp} f \subseteq E$, $\mu(E) < +\infty$. $\{\varphi_n\}_{n \geq 1}$ simple, supported on E, bounded by M, $\varphi_n \to f$ a.e. $\Rightarrow \lim\limits_{n \to \infty} \int_E \varphi_n$ exists. If $f = 0$ a.e., this limit is 0.

Pf Use Egorov. $\forall \varepsilon > 0$, $\exists$ closed $A \subseteq E$ s.t. $\varphi_n \xrightarrow{\text{uniform}} f$ on A and $\mu(E - A) < \varepsilon$. Thus

$$\left| \int_E \varphi_n - \int_E \varphi_m \right| \leq \int_E |\varphi_n - \varphi_m| = \int_A |\varphi_n - \varphi_m| + \int_{E-A} |\varphi_n - \varphi_m|$$

$\leq \int_A |\varphi_n - \varphi_m| + 2M\varepsilon$. Uniform conv. $\overset{\text{Cauchy}}{\Rightarrow}$ $\exists N$, when $m, n > N$, $|\varphi_n - \varphi_m| < \varepsilon$. Thus

when $m,n>N$, $|\int_E \varphi_n - \int_E \varphi_m| \le \mu(A)\varepsilon + 2M\varepsilon$
$\le (\mu(E)+2M)\varepsilon$. $\xRightarrow{\text{Cauchy}} \{\int_E \varphi_n\}$ converges.
If $f=0$ a.e. $|\int_E \varphi_n| \le \int_A |\varphi_n| + 2M\varepsilon$. $\varphi_n \xrightarrow[\text{a.e.}]{\text{uniform}} 0$
$\Rightarrow \exists N$, when $n>N$, $|\varphi_n|<\varepsilon$ a.e.
$\Rightarrow |\int_E \varphi_n| \le (\mu(E)+2M)\varepsilon \Rightarrow \int_E \varphi_n \to 0$ $\square$

☆ $f: \mathbb{R}\to\mathbb{R}$ S2-function. By previous result.
$\exists \{\varphi_n\}_{n\ge1}$ simple bounded, supported on supp$f$.
By the lem, $\lim\limits_{n\to\infty} \int_E \varphi_n$ exists. Define
$\int_{\mathbb{R}} f\,dx = \lim\limits_{n\to+\infty} \int_E \varphi_n$. This is well-defined by the second part of the lem.
For $E \subseteq \mathbb{R}$ measurable, $f\cdot 1_E$ is still S2.
Define $\int_E f\,dx = \int f\cdot 1_E\,dx$.

△ If f is itself simple, this definition coincide with the defn. in Step 1.

☆ (Properties) $f, g : \mathbb{R} \to \mathbb{R}$ S2 functions.

(i) Linearity $\int (af + bg) = a\int f + b\int g$

(ii) Additivity $E \cap F = \phi \Rightarrow \int_{E \cup F} f = \int_E f + \int_F f$

(iii) Monotonicity $f \le g \Rightarrow \int f \le \int g$.

(iv) Triangle Inequality $\left|\int f\right| \le \int |f|$.

(v) $f = 0$ a.e. $\Rightarrow \int f = 0$.

Pf Obvious from defn. $\square$

△ $f \ge 0$ is S2. $\int f = 0 \Rightarrow f = 0$ a.e.

Pf $\forall k \ge 1$ take $E_k = \{x \in \mathbb{R} : f(x) \ge \frac{1}{k}\}$.

Then $f \ge \frac{1}{k} \mathbb{1}_{E_k} \Rightarrow 0 = \int f \ge \frac{1}{k} \mu(E_k)$

$\Rightarrow \mu(E_k) = 0$, $\forall k \in \mathbb{Z}_{>0} \Rightarrow \mu\left(\bigcup_{k=1}^{\infty} E_k\right) = 0$

$\bigcup_{k=1}^{\infty} E_k = \{x \in \mathbb{R} : f(x) > 0\}$ $\square$

Step 3 Non-negative Functions.

☆ $f : \mathbb{R} \to \mathbb{R}_{\ge 0}$ measurable. Define

$$\int_{\mathbb{R}} f\,dx = \sup_{\substack{g \text{ is } S2 \\ 0 \le g \le f}} \left\{ \int g \right\}.$$

If $\int_{\mathbb{R}} f\,dx < +\infty$, we say f is (Lebesgue) integrable.

For $E \subseteq \mathbb{R}$ measurable, $f \cdot \mathbb{1}_E \ge 0$. Define

$$\int_E f\,dx = \int f \cdot \mathbb{1}_E\,dx.$$

★(Properties) $f, g: \mathbb{R} \to \mathbb{R}_{\geq 0}$ measurable.

(i) Linearity $\int(af+bg) = a\int f + b\int g$

(ii) Additivity $E \cap F = \phi \Rightarrow \int_{E\cup F} f = \int_E f + \int_F f$

(iii) Monotonicity $f \leq g \Rightarrow \int f \leq \int g$.

(iv) Triangle Inequality $\left|\int f\right| \leq \int |f|$.

(v) $f = 0$ a.e. $\Rightarrow \int f = 0$.

Pf (i) Clearly $\int af = a\int f$. To see $\int f + \int g = \int(f+g)$, first for $0 \leq \varphi \leq f$, $0 \leq \psi \leq g$, φ, ψ S_2, we have $0 \leq \varphi + \psi \leq f+g$, $\varphi+\psi$ S_2

$\Rightarrow \int(f+g) = \sup_{\eta} \int \eta \geq \int(\varphi+\psi) = \int\varphi + \int\psi$. Take sup gives $\int f + \int g \leq \int(f+g)$. For the other direction, $\forall\, 0 \leq \eta \leq f+g$, η S_2, let $\eta_1 = \min(f, \eta)$, $\eta_2 = \eta - \eta_1$, then η_1, η_2 S_2.

$0 \leq \eta_1 \leq f$, $0 \leq \eta_2 \leq g$

$\Rightarrow \int f + \int g \geq \int \eta_1 + \int \eta_2 = \int \eta$. Take sup.

(ii) ~ (v) obvious. □

△ $f \geq 0$, $\int f = 0 \Rightarrow f = 0$ a.e.

Pf. Same as previous. □

Step 4. General case.

☆ $f:\mathbb{R}\to\mathbb{R}$ measurable. $|f|\geq 0$. Say f is (Lebesgue) integrable if $|f|$ is integrable.

Recall $f^+=\max\{f,0\}$. $f^-=\max\{-f,0\}$, then $0\leq f^{\pm}\leq|f|$, thus integrable. Define

$$\int_{\mathbb{R}} f\,dx=\int_{\mathbb{R}} f^+dx-\int_{\mathbb{R}} f^-dx.$$

△ f integrable. $f=f_1-f_2$. $f_1, f_2\geq 0$ and integrable. Then $\int f=\int f_1-\int f_2$.

Pf $f=f_1-f_2=f^+-f^-\Rightarrow f_1+f^-=f_2+f^+$,

Both sides ≥ 0 integrable, thus

$$\int f_1+\int f^-=\int f_2+\int f^+\Rightarrow \int f_1-\int f_2=\int f^+-\int f^-.$$

☆ (Properties) $f, g:\mathbb{R}\to\mathbb{R}$. measurable.

(i) Linearity $\int(af+bg)=a\int f+b\int g$

(ii) Additivity $E\cap F=\phi\Rightarrow\int_{E\cup F}f=\int_E f+\int_F f$

(iii) Monotonicity $f\leq g\Rightarrow\int f\leq\int g$.

(iv) Triangle Inequality $\left|\int f\right|\leq\int|f|$.

(v) $f=0$ a.e. $\Rightarrow\int f=0$.

Note that (v) means when one modify f on a set of zero measure, both the integrability and the value of the integration remains unchanged.

Summary

Step 1 Simple Func

$f = \sum c_i 1_{E_i}$, $\int f = \sum c_i \mu(E_i)$.

canonical form $\Rightarrow$ well-defined.

Step 2 Bounded Func Sped on Fin. Meas. Set.

Use $\varphi_n \xrightarrow[a.e.]{pt} f$ simple, define $\int f = \lim_{n\to\infty} \int \varphi_n$

Egorov $\Rightarrow$ well-defined.

Step 3 Non-negative Func.

$\int f = \sup_{\substack{0 \le g \le f \\ g \text{ in step } 2}} \{\int g\}$. f integrable if $\int f < +\infty$.

Step 4 General Func.

$f = f^+ - f^-$. $\int f = \int f^+ - \int f^-$

☆ (i) Properties (repeated).

(ii) Observation: one can always modify on a set of meas 0 which does not affect.

(iii) $f \ge 0$, $\int f = 0 \Rightarrow f = 0$ a.e.

☆ (A confusing example) $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} 1_{(n-1,n)}$

Then f is NOT Lebesgue integrable

but Riemann integrable!

2. Convergency Theorems

In Riemann integration theory, if $f_n \xrightarrow{unif} f$ then $\int f_n \to \int f$. But uniform conv. is a very strong condition which is hard to achieve.

△ (A simple example.) $f_n = n 1_{(0,\frac{1}{n})}$

$f_n \xrightarrow{pt.} 0$. $f_n \geq 0$. $\int f_n = 1$.

☆ (Bounded Convergence Thm).

$\{f_n\}$ measurable, bounded by M, spted on E. $\mu(E) < +\infty$, $f_n \xrightarrow[a.e.]{pt.} f$. $\Rightarrow$ f is measurable. bounded. spted on E a.e, and $\int f_n \to \int f$.

Pf Similar to the Lem in Step 2 before.

By Egorov. $\forall \varepsilon > 0$. $\exists A$ s.t. ...

on A. uniform conv $\Rightarrow \exists N$. $n > N$. $|f_n(x) - f(x)| < \varepsilon$ $\forall x \in A$.

Then $\int_E |f_n - f| \leq \int_A |f_n - f| + \int_{E-A} |f_n - f|$.

$$\leq \varepsilon \mu(E) + 2M\varepsilon \quad \square$$

☆ (Fatou's Lemma). $f_n \geq 0$ measurable, $f_n \xrightarrow[a.e]{pt} f$, then $\int f \leq \varliminf_{n\to\infty} \int f_n$.

Pf $\forall\ 0 \leq g \leq f$, g S2. $g_n \overset{def}{=} \min\{g, f_n\}$. $E = \operatorname{spt} g$, $|g| < M$

Then g_n S2. $|g_n| < M$. g_n spted on E.

$g_n \xrightarrow[a.e]{pt} g$. Bounded conv $\Rightarrow$ $\int g_n \to \int g$.

But $g_n \leq f_n$. $\int g_n \leq \int f_n \Rightarrow \varliminf_{n\to\infty} \int g_n \leq \varliminf_{n\to\infty} \int f_n$ □

Rmk: for a general series $f_n \geq 0$ measurable,

$\int \varliminf_{n\to\infty} f_n \leq \varliminf_{n\to\infty} \int f_n$. Pf similar.

△ (Cor) $f \geq 0$. $f_n \geq 0$ measurable $f_n \leq f$, a.e.

$f_n \xrightarrow[a.e.]{pt} f \Rightarrow \int f_n \to \int f$.

Pf First, $f_n \leq f \Rightarrow \int f_n \leq \int f \Rightarrow \varlimsup_{n\to\infty} \int f_n \leq \int f$.

Then by Fatou's lem $\int f \leq \varliminf_{n\to\infty} \int f_n$ □

☆ (Monotone Convergence Thm) $f_n \geq 0$ measurable

$f_n \leq f_{n+1}$, $f_n \xrightarrow[a.e.]{pt} f \Rightarrow \int f_n \to \int f$. Pf By Cor. □

△ (Cor) $a_k \geq 0$ measurable $\int \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \int a_k$.

If RHS $< +\infty$. $\sum_{k=1}^{\infty} a_k$ converges a.e.

The point is that our theory of Lebesgue integration works for extended functions

$f: \mathbb{R} \to [-\infty, +\infty]$. In this sense, obviously

a function is integrable $\Rightarrow$ it is a.e. finite.

△ $f:\mathbb{R}\to\mathbb{R}$ integrable. Then $\forall \varepsilon>0$

(i) $\exists N>0$ s.t. $\int_{\mathbb{R}-[-N,N]}|f|<\varepsilon$.

(ii) $\exists \delta>0$ s.t. $\int_E |f|<\varepsilon$ for $\forall$ measurable E with $\mu(E)<\delta$ (absolutely continuity).

Pf WLOG suppose $f\geq 0$.

(i) $\forall n\in\mathbb{Z}_{>0}$. Let $f_n=f\mathbb{1}_{[-n,n]}$.

Then $0\leq f_n\leq f_{n+1}$. $f_n\xrightarrow{pt} f$. By Monotone conv. thm, $\int f_n\to\int f<+\infty$. By defn of limit. $\exists N$ s.t. $\int f-\int f_N<\varepsilon$.

(ii) $\forall n\in\mathbb{Z}_{>0}$. let $E_n=\{x: f(x)<n\}$

$f_n=f\mathbb{1}_{E_n}$. Then $0\leq f_n\leq f_{n+1}$ $f_n\xrightarrow{pt} f$.

By monotone conv. thm, $\int f_n\to\int f\Rightarrow \exists N$.

$$\int_{\mathbb{R}} f-\int_{E_N} f<\frac{\varepsilon}{2}.$$ Take $\delta=\frac{\varepsilon}{2N}$, then

for $\forall E\subseteq\mathbb{R}$ with $\mu(E)<\delta$.

$$\int_E f=\int_{E_N} f+\int_{E-E_N} f\leq N\mu(E_N)+\int_{\mathbb{R}-E_N} f$$

$$\leq\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$ □

☆ (Dominated Conv. Thm) f_n measurable $f_n \xrightarrow{a.e.} f$. If $\exists g$ integrable s.t. $|f_n| \le g$, then $\int f_n \to \int f$.

Pf $\forall n \in \mathbb{Z}_{>0}$ let $E_n = \{x : |x| \le n,\ g(x) \le n\}$.

Take N s.t. $\int_{E_N^c} g < \varepsilon$ $(g 1_{E_n} \nearrow g)$.

Then $f_n 1_{E_N}$ are bounded by N, sp'ted on E_N. bounded conv. thm $\Rightarrow \int_{E_N} |f_n - f| \to 0$. Find N_0 s.t. $n > N_0 \Rightarrow \int_{E_N} |f_n - f| < \varepsilon$. Then when $n > N_0$,

$$\int |f_n - f| = \int_{E_N} |f_n - f| + \int_{E_N^c} |f_n - f|$$
$$\le \varepsilon + \int_{E_N^c} 2g \le 3\varepsilon. \qquad \square$$