$(X, \Sigma, \mu)$ measure space

(e.g. $\mu^*$ outer measure on $X$, $\Sigma$ = set of $\mu^*$-measurable sets in $X$)

Assume $\mu(X) < +\infty$

$f : X \to \mathbb{R}$

$f : [a,b] \to \mathbb{R} \leftarrow$ Riem. integrals

$P : a = x_0 < x_1 < \cdots < x_n = b$

$S(P, \{x_i^*\}, f) := \sum_i f(x_i^*)(x_i - x_{i-1})$, $x_i^* \in [x_{i-1}, x_i]$

$X = \underbrace{\coprod_i X_i}_{\text{partition}}$, $X_i \in \Sigma$

$S(\coprod_i X_i, \{x_i^*\}, f) := \sum_i f(x_i^*)\, \mu(X_i)$, $x_i^* \in X_i$

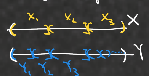
$f : X \to \mathbb{R}$ is Lebesgue integrable over $X$

$\overset{\text{def}}{\Leftrightarrow}$ $\exists I \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$, $\exists P = \{\coprod_i X_i\}$ ($X_i \in \Sigma$, $\coprod_i X_i = X$) s.t.

$Q = \coprod_j Y_j \overset{\text{refinement of}}{\succeq} P \Rightarrow |S(Q, \{y_j^*\}, f) - I| < \varepsilon$.

$I := \int_X f \, d\mu$.

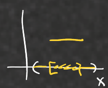
Def: $\coprod_j Y_j \succeq \coprod_i X_i \overset{\text{def}}{\Leftrightarrow} \forall j,\ Y_j \subset X_i\ \exists i$.



e.g. $\chi_A : X \to \mathbb{R}$, $A \subset X$, $A \in \Sigma$. $\mu(X) < \infty$

$\chi_A = \mathbb{1}_A$

$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$



$P = \{ \underset{①}{A} \amalg \underset{②}{(X-A)} \}$, $x_1^* \in A$, $x_2^* \in X - A$

$S(P, \chi_A) = \underset{\chi_A(x_1^*)}{1} \cdot \mu(A) + \underset{\chi_A(x_2^*)}{0} \cdot \mu(X-A) = \mu(A)$.

$Q = \{\coprod_j Y_j\} \succeq P$. $y_j^* \in Y_j$

$S(Q, \chi_A) = \sum_j \chi_A(y_j^*)\mu(Y_j) = \sum_{Y_j \subset A} 1 \cdot \mu(Y_j) + \cancel{\sum_{Y_j \subset X-A} 0 \cdot \mu(Y_j)}$

$= \sum_{Y_j \subset A} \mu(Y_j) = \mu(\underbrace{\coprod_{Y_j \subset A} Y_j}_{\cap\, A\ \ \cup\, A}) = \mu(A)$

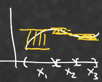
$\Rightarrow |S(Q, \chi_A) - \mu(A)| = 0$

$\therefore$ $\chi_A$ is Lebesgue integrable and $\int_X \chi_A \, d\mu = \mu(A)$.

Prop: $f : X \to \mathbb{R}$ is Lebesgue integrable on $X$

$\Leftrightarrow \forall \varepsilon > 0$, $\exists P = \{\coprod_i X_i\}$ of $X$ s.t.

$\underbrace{\omega(P, f)}_{= \sum_i (\sup_{X_i} f - \inf_{X_i} f)\mu(X_i)} < \varepsilon$



Proof: ($\Rightarrow$) easy. $-\frac{\varepsilon}{2} < S(P, \{x_i^*\}, f) - I < \varepsilon/2$.

(Note: $\coprod_i Y_i \underset{\text{refinement}}{\succeq} \coprod_i X_i$ forms a directed set.)

$\coprod_{i,j} (X_i \cap Y_j)$ $\quad X_i \cap Y_j$ $(i,j)$ $\quad X_k \cap Y_\ell$ $(k,\ell)$

$S(\coprod X_i, \{x_i^*\}, f) : (\mathcal{D}, \succeq) \to \mathbb{R} \leftarrow$ complete.

($\Leftarrow$)

$|S(\coprod_j Y_j, f) - S(\coprod_k Z_k, f)|$

$\le \underbrace{|S(\coprod_j Y_j, f) - S(\coprod_i X_i, f)|} + |S(\coprod_i X_i, f) - S(\coprod_k Z_k, f)|$

$\forall \varepsilon > 0$. $\exists P = \{\coprod_{i=1}^{\infty} X_i\}$ s.t. $\omega(P, f) < \varepsilon/2$.

$\coprod_j Y_j \succeq \coprod_i X_i$

$\Rightarrow |S(\coprod_j Y_j, f) - S(\coprod_i X_i, f)|$

$= |\sum_j f(y_j^*)\mu(Y_j) - \sum_i f(x_i^*)\mu(X_i)|$, $y_j^* \in Y_j$, $x_i^* \in X_i$

$= |\sum_i \sum_{Y_j \subset X_i} f(y_j^*)\mu(Y_j) - \sum_i f(x_i^*)\mu(X_i)|$, $X_i = \coprod_{Y_j \subset X_i} Y_j$

$Y_j$'s. $X_i$

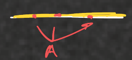
$= |\sum_i (\sum_{Y_j \subset X_i} f(y_j^*)\mu(Y_j)) - \sum_i f(x_i^*) \sum_{Y_j \subset X_i} \mu(Y_j)|$

$= |\sum_i \sum_{Y_j \subset X_i} (f(\underset{\in X_i}{y_j^*}) - f(\underset{\in X_i}{x_i^*}))\mu(Y_j)|$

$\le \sup_{X_i} f - \inf_{X_i} f$

$\le \sum_i \underbrace{\boxed{\sum_{Y_j \subset X_i}} (\sup_{X_i} f - \inf_{X_i} f) \boxed{\mu(Y_j)}}_{\mu(X_i)} = \omega(P, f) < \varepsilon/2$, $\coprod_i X_i$

$f = g$ a.e. (almost everywhere) on $X$ $\overset{\text{def}}{\Leftrightarrow}$ $\exists A \in \Sigma$, $\mu(A) = 0$ s.t. $f = g$ on $X - A$.



Properties:

① $f = g$ a.e. on $X$ and $f$ is Lebesgue integrable

then: $g$ is Lebesgue integrable.

$S(\coprod_j Y_j, f) = \sum_{Y_j \subset A} f(y_j^*) \mu(Y_j) + \sum_{Y_j \subset X-A} f(y_j^*)\mu(Y_j)$, $\mu(Y_j) = 0$, $f(y_j^*) = g(y_j^*)$; $f = g$ on $X - A$ where $\mu(A) = 0$.

$P = \coprod_{i=0}^{\infty} X_i = A \amalg (\coprod_{i=1}^{\infty} X_i)$

$= S(\coprod_j Y_j, g)$.

② ✓

③ $f, g$ Lebesgue integrable on $X$, $f \ge g$ a.e. on $X$.

then $\int_X f \, d\mu \ge \int_X g \, d\mu$. $(h := f - g)$

If furthermore: $\int_X f \, d\mu = \int_X g \, d\mu$, $\Rightarrow f = g$ a.e. on $X$.

Proof (of equality part)

WANT: $h := f - g = 0$ a.e. on $X$. ($f \ge g$ a.e.) $f \ge g$ on $X - A$

$\int_X h \, d\mu = \int_X f \, d\mu - \int_X g \, d\mu = 0$.

$\forall \varepsilon > 0$, $\exists P_\varepsilon = \{\coprod_i X_i^\varepsilon\}$ s.t. $|S(\coprod_i X_i^\varepsilon, h)| < \varepsilon$.

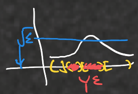
WLOG assume one of $X_i^\varepsilon = A$.

$\sum_i h(x_i^*) \mu(X_i^\varepsilon) < \varepsilon$. $x_i^* \in X_i^\varepsilon$

$\Rightarrow \sum_i (\sup_{X_i^\varepsilon} h)\mu(X_i^\varepsilon) \le \varepsilon$.

$\sum_{\{\sup_{X_i^\varepsilon} h \ge \sqrt{\varepsilon}\}} \sqrt{\varepsilon}\mu(X_i^\varepsilon) \le \Big( \underbrace{\sum_{\{\sup_{X_i^\varepsilon} h \ge \sqrt{\varepsilon}\}}} + \underbrace{\sum_{\{\sup_{X_i^\varepsilon} h < \sqrt{\varepsilon}\}}}_{+ \text{ or } 0} \Big) (\sup_{X_i^\varepsilon} h)\, \mu(X_i^\varepsilon) \le \varepsilon$.

$Y^\varepsilon := \bigcup_{\{\sup_{X_i^\varepsilon} h \ge \sqrt{\varepsilon}\}} X_i^\varepsilon$



$\sqrt{\varepsilon}\, \mu(Y^\varepsilon) \le \varepsilon \Rightarrow \mu(Y^\varepsilon) \le \sqrt{\varepsilon}$.

If $x \in X - Y^\varepsilon \Rightarrow h(x) < \sqrt{\varepsilon}$.

take $\varepsilon_n \to 0^+$. If $x \in$ infinitely many of $(X - Y^{\varepsilon_n})$

$\Rightarrow h(x) < \sqrt{\varepsilon_n}$ for $\infty$-many $n$.

$\Rightarrow h(x) = 0$.

$\bigcap_{k=1}^{\infty} \bigcup_{n \ge k} (X - Y^{\varepsilon_n}) = X - \bigcup_{k=1}^{\infty} \underbrace{\bigcap_{n \ge k} Y^{\varepsilon_n}}_{\text{monotone}}$

$\mu(\bigcup_{k=1}^{\infty} \bigcap_{n \ge k} Y^{\varepsilon_n}) = \lim_{k \to \infty} \mu(\bigcap_{n \ge k} Y^{\varepsilon_n}) \le \lim_{k \to \infty} \underbrace{\mu(Y^{\varepsilon_k})}_{< \sqrt{\varepsilon_k} \to 0}$.

$x \in X - \bigcup_{k=1}^{\infty} \bigcap_{n \ge k} Y^{\varepsilon_n}$

$\Rightarrow h(x) = 0$.

$h = 0$ a.e. on $X$.
($h = 0$ on $X - \bigcup_{k=1}^{\infty} \bigcap_{n \ge k} Y^{\varepsilon_n}$)

Def: $f : X \to \mathbb{R}$ is said to be measurable

$\overset{\text{def}}{\Leftrightarrow}$ $f^{-1}([a,b))$ is $\underbrace{\text{measurable}}_{\in \Sigma}$ $\forall a, b$.

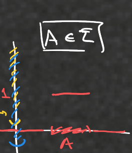
Note $(a,b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b)$

$f^{-1}((a,b)) = \underbrace{\bigcup_{n=1}^{\infty} \underbrace{f^{-1}([a+\frac{1}{n}, b))}_{\in \Sigma}}_{\in \Sigma}$

$(a,b] = \mathbb{R} \setminus ((-\infty, a] \cup (b, \infty))$

e.g. $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$ $\boxed{A \in \Sigma}$

$\chi_A^{-1}((a, \infty)) = \begin{cases} \emptyset & \text{if } a \ge 1 \\ A & \text{if } a \in [0,1) \\ X & \text{if } a < 0 \end{cases}$



Claim: $\begin{cases} g : \mathbb{R} \to \mathbb{R} & \text{continuous} \\ f : X \to \mathbb{R} & \text{measurable} \end{cases}$

$\Rightarrow g \circ f : X \to \mathbb{R}$ is measurable.

$(g \circ f)^{-1}((a,b)) = f^{-1}(\underbrace{g^{-1}((a,b))}_{\text{cts, open } = \coprod_{i=1}^{\infty}(\ )}) = \coprod_{i=1}^{\infty} \underbrace{f^{-1}((\ \ ))}_{\text{measurable}}$

$f^2$ $\cos(f)$, $\sin(f)$, $e^f$

Claim: $f, g : X \to \mathbb{R}$ measurable

then $f+g$, $f-g$, $fg$ are measurable.

Proof: $x \in (f+g)^{-1}((a, \infty))$

$\Leftrightarrow f(x) + g(x) > a$

$\Leftrightarrow f(x) > a - g(x)$

$\Leftrightarrow \exists q \in \mathbb{Q}$ s.t. $f(x) > q > a - g(x)$

$\Leftrightarrow x \in \underbrace{\bigcup_{q \in \mathbb{Q}} f^{-1}((q, +\infty)) \cap g^{-1}((a-q, +\infty))}_{= (f+g)^{-1}((a,\infty))}$ ✓

$\therefore$

$fg = \frac{1}{4}\big( \underbrace{(f+g)^2}_{s \circ (f+g)} - \underbrace{(f-g)^2}_{s \circ (f-g)} \big)$ ▨

$s(x) = x^2$

$x \in (\frac{1}{f})^{-1}((a, \infty)) \Leftrightarrow \frac{1}{f(x)} > a \Leftrightarrow \begin{cases} a > 0 \\ f(x) > 0 \\ f(x) < \frac{1}{a} \end{cases}$ or $\begin{cases} a < 0 \\ f(x) > 0 \end{cases}$

or $\begin{cases} a > 0 \\ \\ \end{cases}$ $\begin{cases} a < 0 \\ \\ \end{cases}$

or $\begin{cases} a = 0 \end{cases}$ $\begin{cases} a = 0 \end{cases}$

Claim: $f_n : X \to \mathbb{R}$ measurable $\forall n$.

$\Rightarrow \inf_n f_n(x)$ and $\sup_n f_n(x)$ are measurable.

Proof: $g(x) := \sup_n f_n(x)$

$x \in g^{-1}((-\infty, a])$

$\Leftrightarrow g(x) \le a$

$\Leftrightarrow \sup_n f_n(x) \le a$

$\Leftrightarrow f_n(x) \le a \quad \forall n$.

$\Leftrightarrow x \in \underbrace{\bigcap_{n=1}^{\infty} \underbrace{f_n^{-1}((-\infty, a])}_{\in \Sigma}}_{\in \Sigma}$

$g^{-1}((-\infty, a]) = \bigcap_{n=1}^{\infty} f_n^{-1}((-\infty, a])$

Cor: $\underbrace{\liminf_{n \to \infty} f_n(x)}_{= \sup_k \inf_{n \ge k} f_n(x)}$ and $\underbrace{\limsup_{n \to \infty} f_n(x)}_{= \inf_k \sup_{n \ge k} f_n(x)}$ are measurable.

Cor[2] If $\lim_{n \to \infty} f_n(x)$ exists $\forall x \in X$.

then $f_\infty(x) = \lim_{n \to \infty} f_n(x)$ is measurable.

Cor[3] If $\sum_{n=1}^{\infty} f_n(x)$ converges $\forall x \in X$.

$\Rightarrow F(x) := \sum_{n=1}^{\infty} f_n(x) = \underbrace{\lim_{N \to \infty} \underbrace{\sum_{n=1}^{N} f_n(x)}_{✓}}_{✓}$

Theorem: $\mu(X) < \infty$, $f$ bounded.

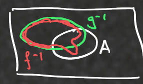
$f : X \to \mathbb{R}$ is Lebesgue integrable on $X$

$\Leftrightarrow f = g$ a.e. where $g$ is measurable.

$f = g$ on $X - A$, $\mu(A) = 0$. ($g$ measurable)   Q: $f$ is measurable?

$\boxed{f^{-1}((a,\infty))}$ $\boxed{g^{-1}((a,\infty))}$

$f(x) = g(x)$ if $x \in X - A$.

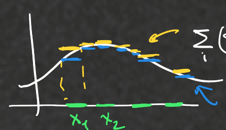


$f^{-1}((a,\infty)) = (g^{-1}((a,\infty)) \cap (X - A)) \cup (\underbrace{f^{-1}((a,\infty)) \cap A}_{\in \Sigma})$ Not always!

($\Rightarrow$) $\forall \varepsilon_n > 0$, $\exists P^{(n)} = \{\coprod_i X_i^{(n)}\}$ s.t.

$\omega(P^{(n)}, f) < \varepsilon_n$.

$\sum_i (\sup_{X_i^{(n)}} f)\mu(X_i^{(n)}) - \sum_i (\inf_{X_i^{(n)}} f)\mu(X_i^{(n)}) < \varepsilon_n$



$\sum_i (\sup_{X_i^{(n)}} f)\chi_{X_i^{(n)}}(x) = \phi_n(x)$
$\sum_i (\inf_{X_i^{(n)}} f)\chi_{X_i^{(n)}}(x) = \psi_n(x)$
simple functions

$\psi_n \le f \le \phi_n \quad \forall x \in X$.

$g := \underbrace{\sup_n \psi_n}_{\text{measurable}} \le f \le \inf_n \phi_n = h \quad \forall x \in X$.

$0 \le \int_X (g - h) \le \int_X (\phi_n - \psi_n) = \underbrace{\sum_i (\sup_{X_i^{(n)}} f - \inf_{X_i^{(n)}} f)\mu(X_i^{(n)})}_{< \varepsilon_n \to 0}$

$\Rightarrow \begin{cases} \int_X g = \int_X h \Rightarrow & g = h \text{ a.e.} \\ g \le h & \Rightarrow g = f = h \text{ a.e.} \end{cases}$