Tutorial 5 Problems

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Recollections.

Last week you learned

- some abstract measure theory.
- set operations. Recall the following important facts in any complete measure space:

Countably Additivity. If $\{E_k\}$ is a series of mutually disjoint measurable sets, then $\cup_{k=1}^{\infty} E_k$ is measurable and

$$\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k)$$

Increasing & Decreasing series of sets. Let $\{E_k\}$ be a series of measurable sets.

(i) If $E_1 \subseteq E_2 \subseteq \cdots$, then $\cup_{k=1}^{\infty} E_k$ is measurable and

$$\mu(\bigcup_{k=1}^{\infty} E_k) = \lim_{k\to\infty} \mu(E_k)$$

(ii) If $E_1 \supseteq E_2 \supseteq \cdots$, then $\cap_{k=1}^{\infty} E_k$ is measurable and

$$\mu(\bigcap_{k=1}^{\infty} E_k) = \lim_{k\to\infty} \mu(E_k)$$

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1 Set Operations.

Let $\{A_k\}$ be a series of sets. Define

$$\limsup A_k = \bigcap_{j=1}^{\infty} \bigcup_{k \geq j} A_k, \liminf A_k = \bigcup_{j=1}^{\infty} \bigcap_{k \geq j} A_k$$

Show that

(i) $\limsup A_k = \{x : \forall j, \exists k \geq j \text{ s.t. } x \in A_k\}$ is the set of x that lies in infinitely many A_k's.

(ii) $\liminf A_k = \{x : \exists j, \text{ s.t. } \forall k \geq j, x \in A_k\}$ is the set of x that lies in almost all A_k's.

Remark 1.0.1. *Here and after "almost all" means "all but finitely many". Intuitively,* $\cap$ *means "for all", and* $\cup$ *means "exists".*

Slogan: $\boxed{\text{Do approximations.}}$

Examples.

- Let $f : \mathbf{R} \to \mathbf{R}$ be a function. Then

$$\{x \in \mathbf{R} : f(x) > 0\} = \bigcup_{k=1}^{\infty} \{x \in \mathbf{R} : f(x) \geq \frac{1}{k}\}$$

- Let $f : \mathbf{R} \to \mathbf{R}$ be a differentiable function. Then $\forall a \in \mathbf{R}$,

$$\{x \in \mathbf{R} : f'(x) > a\} = \bigcup_{k=1}^{\infty} \bigcap_{n \geq k} \{x \in \mathbf{R} : f(x + \frac{1}{n}) - f(x) \geq \frac{a}{n}\}$$

  This will be used to prove the measurability of $f'(x)$ later.

- Let $\{f_k\}, f : \mathbf{R} \to \mathbf{R}$ be functions. Then the set of $x \in \mathbf{R}$ s.t. $\{f_k(x)\}$ does not converge to $f(x)$ is

$$\bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n \geq N} \{x \in \mathbf{R} : |f_n(x) - f(x)| \geq \frac{1}{k}\}$$

  This will be used to prove Egorov's theorem later.

2 Borel Sets.

Philosophy: $\boxed{\text{Measurable sets are almost Borel.}}$

To be precise, let $E \subseteq \mathbf{R}$ be a measurable set. Show that there exists a G_δ-set $H \supseteq E$ such that $\mu(H - E) = 0$, there exists a F_σ-set $K \subseteq E$ such that $\mu(E - K) = 0$.

Regularity of Outer Measure. $\forall E \subset \mathbf{R}, \exists G_\delta$-set $H \supseteq E$ s.t. $\mu(H) = \mu^*(E)$. There is a similar version for inner measures.

Discontinuous points of a function. Let $f : \mathbf{R} \to \mathbf{R}$ be a function. The goal of this problem is to prove that the set of discontinuous points of f is an F_σ-set.

(i) For $x \in \mathbf{R}$, define the **oscillation** of f near x by $o_f(x) = \inf\{D(f(U)) : U \text{ open nbhd. of } x\}$. (Here for any set $S \subset \mathbf{R}$, define the **diameter** of S by $D(S) = \sup_{x,y\in S} |x - y|$.) Show that f is continuous at x iff. $o_f(x) = 0$.

(ii) Show that for any $\epsilon > 0$, the set $E_\epsilon = \{x \in \mathbf{R} : o_f(x) > \epsilon\}$ is open.

Let C be the set of continuous points of f, then $C = \cap_{n=1}^{\infty} E_{\frac{1}{n}}$ is G_δ, thus the set of discontinuous points is F_σ.

3 Translation Invariance.

Let $E \subseteq \mathbf{R}$ be a set. For any $a \in \mathbf{R}$ define $E + a = \{x + a : x \in E\}$. The goal of this section is to prove that if E is measurable, then $\mu(E) = \mu(E+a)$, namely the Lebesgue measure on $\mathbf{R}$ is invariant under translations. This is an example of **Haar measure**. The idea of the proof is to use approximations, justifying our slogan.

Case of intervals. If E is an open interval, so is $E + a$. Obviously they have the same length.

Case of open sets. If E is an open set, by the structure theorem of open sets in $\mathbf{R}$, E is a disjoint union of open intervals, and the result follows from the case of intervals.

Outer measure is invariant under translations. This follows from the definition of outer measure and the case of open sets. In particular, this implies that the translation of a zero measure set is still a zero measure set.

Case of a G_δ-set. This follows from the case of open sets.

Case of a general measurable set. By the result above, there exists a G_δ-set H and a zero measure set Z s.t. $H = E \sqcup Z$. Then the result follows from the case of G_δ set and zero measure set.

Remark 3.0.1. *In this proof, we use a series of approximations*
$\boxed{\textit{Intervals}} \to \boxed{\textit{Open Sets}} \to \boxed{G_\delta\textit{-sets}} \to \boxed{\textit{General Measurable Set}}$
to reduce to the case of intervals, which is obvious.

4 The Boss

Find a measurable set which is **NOT** a Borel set.