μ^* = Lebesgue outer measure

① $\mu^*(\emptyset)=0$

② $A\subset B\subset\mathbb{R}$, $\mu^*(A)\le\mu^*(B)$.

③ $\mu^*(\bigcup_{i=1}^{\infty}A_i)\le\sum_{i=1}^{\infty}\mu^*(A_i)$

Carathéodory's Theorem

$\mu^*(A)=\mu_*(A)$ (bounded) $\iff$ $\forall X\subset\mathbb{R}$ (bounded), $\mu^*(X)=\mu^*(X\cap A)+\mu^*(X-A)$ $(*)$

Lemma: $A\cap B=\emptyset$

$\mu_*(A\amalg B)\le\mu_*(A)+\mu^*(B)\le\mu^*(A\amalg B)$

Proof of Carathéodory's Theorem:

$(\Leftarrow)$ $A\subset \mathcal{U}$ (open, bounded.)

$\mu^*(\mathcal{U})=\mu^*(\overset{A}{\mathcal{U}\cap A})+\mu^*(\mathcal{U}-A)$ (from $(*)$).

$\mu_*(\mathcal{U})=\mu_*((\mathcal{U}-A)\amalg A)\le\mu^*(\mathcal{U}-A)+\mu_*(A)$

$\mu^*(A)\le\mu_*(A)\le\mu^*(A)\Rightarrow\mu_*(A)=\mu^*(A)$.

$(\Rightarrow)$ Assume $\mu_*(A)=\mu^*(A)$

WANT: $\mu^*(\underset{\text{arbitrary}}{X})\ge\mu^*(X\cap A)+\mu^*(X-A)$.

$\forall\varepsilon>0$, $\exists\,\mathcal{U}\supset X$ s.t. $\mu^*(X)\le\mu(\mathcal{U})<\mu^*(X)+\varepsilon$.

$(\mathcal{U}\cap A)\amalg(\mathcal{U}-A)=\mathcal{U}$ — $\mu^*(\mathcal{U}\cap A)+\mu^*(\mathcal{U}-A)$ $\ge$ $\mu^*(X\cap A)+\mu^*(X-A)$.

$X^{\neq\emptyset}$ set, $\mu^*:\mathcal{P}(X)\to[0,\infty]$ satisfies: $\mu^*(\text{subset of }X)\in[0,\infty]$.

① $\mu^*(\emptyset)=0$

② $A\subset B\subset X\Rightarrow\mu^*(A)\le\mu^*(B)$.

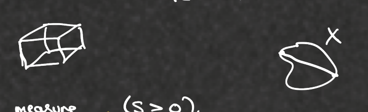
③ $\mu^*(\bigcup_{i=1}^{\infty}A_i)\le\sum_{i=1}^{\infty}\mu^*(A_i)$. $A_i\subset X$.

Then, call μ^* an outer measure on X.

Define: A is μ^*-measurable $\overset{\text{def}}{\iff}$ $\forall Y\subset X$ we have $\mu^*(Y)=\mu^*(Y\cap A)+\mu^*(Y-A)$

e.g. $\mathbb{R}^n$. n-dim Lebesgue outer measure:

$\mu^*(A):=\inf\{\sum_{i=1}^{\infty}\mathrm{vol}(C_i): A\subset\bigcup_{i=1}^{\infty}C_i\}$



e.g. $\mathbb{R}^n$ Hausdorff measure $(s>0)$.

$H^s_\delta(A):=\inf\{\sum_{i=1}^{\infty}\left(\frac{\mathrm{diam}X_i}{2}\right)^s : A\subset\bigcup_{i=1}^{\infty}X_i,\ \mathrm{diam}(X_i)<\delta\}$, $A\subset\mathbb{R}^n$

$H^s(A):=\sup_{\delta>0}H^s_\delta(A)$.

s-dim Hausdorff measure.

If: $\infty>H^s(A)\neq0$, $u<s<t$

then: $H^u(A)=+\infty$, $H^t(A)=0$.

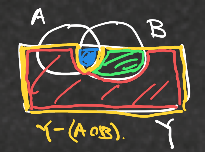
s := Hausdorff dim. of A.

Geometric Measure Theory.

μ^* outer measure on X

If A,B are μ^*-measurable, then so do $A\cup B$, $A\cap B$ and $A-B$.

$A\cap B$: $\forall Y\subset X$: $\mu^*(\underset{\text{any}}{Y})=\mu^*(Y\cap A)+\mu^*(Y-A)$ (B, $-B$.)



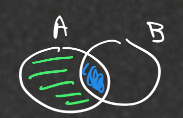
$\mu^*(Y\cap A\cap B)+\mu^*(Y-(A\cap B))$

$=\mu^*(Y\cap A\cap B)+\mu^*(\ \ )$ $Y-B$

$+\mu^*(\ \ )$ $(B-A)\cap Y = Y\cap B\cap A^c = Y\cap B-A$

$=\mu^*(Y\cap B)+\mu^*(Y-B)=\mu^*(Y)$.

$\mu^*(A\cup B)=\mu^*(A)+\mu^*(B)-\mu^*(A\cap B)$. (measurable)



Proof: $\mu^*(\underbrace{A\cup B}_{Y})=\mu^*(\underbrace{(A-B)\amalg B}_{Y})$

$=\mu^*(Y\cap B)+\mu^*(Y-B)$

$=\mu^*(B)+\mu^*(A-B)$.

$\mu^*(A)-\mu^*(A\cap B)$

Cor: $\underbrace{A_1\amalg\cdots\amalg A_N}_{\text{disjoint}}\Rightarrow\mu^*(A_1\amalg\cdots\amalg A_N)=\mu^*(A_1)+\cdots+\mu^*(A_N)$.

Prop: $\{A_i\}_{i=1}^{\infty}$ μ^*-measurable, $A_i\cap A_j=\emptyset$ $(i\neq j)$.

then $\bigcup_{i=1}^{\infty}A_i$ is μ^*-measurable and

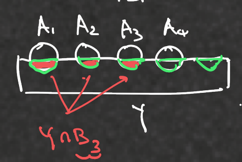
$\mu^*(\bigcup_{i=1}^{\infty}A_i)=\sum_{i=1}^{\infty}\mu^*(A_i)$ (countable additivity)

Proof: $B_n:=\bigcup_{i=1}^{n}A_i$ measurable

$\forall Y\subset X$. $\mu^*(Y)=\mu^*(Y\cap B_n)+\mu^*(Y-B_n)$

$A:=\bigcup_{i=1}^{\infty}A_i$. $\mu^*(Y\cap A)=\mu^*(Y\cap A\cap B_n)$ $+\mu^*(Y\cap A-B_n)$

$=\mu^*(Y\cap B_n)+\mu^*(Y\cap\bigcup_{i>n}A_i)$



$\mu^*(Y-A)\le\mu^*(Y-B_n)$

$\Rightarrow\mu^*(Y\cap A)+\mu^*(Y-A)$

$\le\mu^*(Y\cap B_n)+\mu^*(Y-B_n)+\mu^*(Y\cap\bigcup_{i>n}A_i)$

$=\mu^*(Y)+\mu^*(\bigcup_{i>n}(Y\cap A_i))$

$\le\sum_{i>n}\mu^*(Y\cap A_i)\to0$ (if $\mu^*(Y)<\infty$)

WANT: $\sum_{i=1}^{\infty}\mu^*(Y\cap A_i)<+\infty$.

$\sum_{i=1}^{N}\mu^*(Y\cap A_i)=\mu^*(\bigcup_{i=1}^{N}(Y\cap A_i))=\mu^*((\bigcup_{i=1}^{N}A_i)\cap Y)\le\mu^*(Y)$.

Need: $\mu^*(Y)\ge\mu^*(Y\cap A)+\mu^*(Y-A)$.

if $=\infty$, then done!

$\mu^*(A)=\mu^*(A\cap B_n)+\mu^*(A-B_n)$ $\quad$ $A=\bigcup_{i=1}^{\infty}A_i$

$=\mu^*(B_n)+\mu^*(\bigcup_{i>n}A_i)$ $\quad$ $B_n=\coprod_{i=1}^{n}A_i$

$=\mu^*(A_1)+\cdots+\mu^*(A_n)+$

$\mu^*(\bigcup_{i>n}A_i)\le\underbrace{\sum_{i>n}\mu^*(A_i)}$

$\downarrow 0$ IF $\mu^*(A)<\infty$. DONE

If $\mu^*(A)<\infty$,

$\sum_{i=1}^{N}\mu^*(A_i)=\mu^*(\coprod_{i=1}^{N}A_i)\le\mu^*(A)<\infty$.

Case: $\mu^*(A)=\infty$. $\quad$ $\underbrace{\mu^*(\bigcup_{i=1}^{\infty}A_i)}_{\infty}\le\underbrace{\sum_{i=1}^{\infty}\mu^*(A_i)}_{\infty}$. $\Rightarrow$ $\therefore\ \cdot\cdot=\cdot\cdot$

X, $\Sigma\subset\mathcal{P}(X)$

① $X\in\Sigma$

② $A,B\in\Sigma\Rightarrow A-B\in\Sigma$

③ $A_i\in\Sigma$, $i\in\mathbb{N}\Rightarrow\bigcup_{i=1}^{\infty}A_i\in\Sigma$.

then call Σ: sigma-algebra, σ-algebra

e.g. μ^* outer measure on X

Σ := "set of measurable subsets of X" is a σ-algebra on X.

e.g. $\mathcal{P}(X)$

e.g. $\{\emptyset, X\}$.

e.g. $\mathcal{C}$ = set of all closed sets in X. $\leftarrow$ metric space.

$\Sigma(\mathcal{C})$ = the smallest sigma-algebra $\supset\mathcal{C}$.

$=\bigcap_{\Sigma\supset\mathcal{C}}\Sigma$ $\leftarrow$ Borel σ-algebra.

A set is in F_σ $\overset{\text{def}}{\iff}$ countable union of closed sets

. . . G_δ $\overset{\text{def}}{\iff}$ countable intersection of open sets.

Prop: Given $A_1\subset A_2\subset A_3\subset\cdots$ all measurable.

$\mu^*(\bigcup_{i=1}^{\infty}A_i)=\lim_{n\to\infty}\mu^*(A_n)$.

Proof: $\bigcup_{i=1}^{\infty}A_i=A_1\amalg(A_2-A_1)\amalg(A_3-A_2)\amalg\cdots\amalg(A_N-A_{N-1})\amalg\cdots$

$\mu^*(\bigcup_{i=1}^{\infty}A_i)=\mu^*(A_1)+\sum_{i=1}^{\infty}\underbrace{\mu^*(A_{i+1}-A_i)}_{=\mu^*(A_{i+1})-\mu^*(A_i)}$

$=\mu^*(A_1)+\lim_{N\to\infty}\underbrace{\sum_{i=1}^{N}(\mu^*(A_{i+1})-\mu^*(A_i))}_{\mu^*(A_N)-\mu^*(A_1)}$

$=\lim_{N\to\infty}\mu^*(A_N)$.

Prop: $A_1\supset A_2\supset A_3\supset\cdots$ all μ^*-measurable.

$A:=\bigcap_{n=1}^{\infty}A_n$.

then if $\mu^*(A_1)<+\infty$, then

$\mu^*(A)=\lim_{n\to\infty}\mu^*(A_n)$.

Proof: $B_n:=A_1-A_n$.

$B_1\subset B_2\subset B_3\subset\cdots$

$\mu^*(\bigcup_{n=1}^{\infty}B_n)=\lim_{n\to\infty}\mu^*(B_n)=\lim_{n\to\infty}(\mu^*(A_1)-\mu^*(A_n))$

$\mu^*(\bigcup_{n=1}^{\infty}(A_1-A_n))$ $\quad=\cancel{\mu^*(A_1)}-\lim_{n\to\infty}\mu^*(A_n)$

$\mu^*(A_1-\bigcap_{n=1}^{\infty}A_n)=\cancel{\mu^*(A_1)}-\mu^*(\bigcap_{n=1}^{\infty}A_n)$.

$B=\bigcap_{k=1}^{\infty}\underbrace{\bigcup_{n>k}A_n}_{E_k}=(A_2\cup A_3\cup A_4\cup\cdots)\cap(A_3\cup A_4\cdots)\cap(A_4\cup\cdots)$

$C=\bigcup_{k=1}^{\infty}\bigcap_{n>k}A_n$ $\quad$ $\boxed{B\supset C}$

B = set of $x\in X$ s.t. $x\in$ infinitely many A_i's.

$x\in B\Rightarrow x\in E_1\Rightarrow\exists n_1>1$ s.t. $x\in A_{n_1}$.

$x\in E_{n_1}\Rightarrow\exists n_2>n_1$ s.t. $x\in A_{n_2}$.

$x\in E_{n_2}$ = — — .

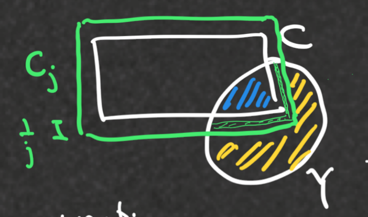
C = set of $x\in X$ s.t. x is eventually in $A_n, A_{n+1}, A_{n+2},\ldots$

HW: If $\sum_{i=1}^{\infty}\mu^*(A_i)<+\infty$.

then $\mu^*(B)=0$.

$\mu^*(\bigcap_{n=1}^{\infty}\underbrace{\bigcup_{k>n}A_k}_{E_n})$

Borel–Cantelli's lemma



$C_j:=\{x: d(x,C)\le\frac{1}{j}\}$.

want: $\mu^*(Y\cap C)+\mu^*(Y-C)=\mu^*(Y)$

$\mu^*(Y\cap C)+\mu^*(Y-C_j)=\mu^*((Y\cap C)\cup(Y-C_j))$ $\le\mu^*(Y)$

(X,Σ,μ) $\leftarrow$ measure space. Σ: σ-algebra. $\quad$ $\mu:\Sigma\to[0,\infty]$. is an abstract measure

if

① $\mu(\emptyset)=0$

② $\mu(A)\ge0$ $\quad\forall A\in\Sigma$.

③ $A_i\in\Sigma$ $(i=1,2,3,\ldots)$, $A_i\cap A_j=\emptyset$ $(i\neq j)$

$\Rightarrow\mu(\coprod_{i=1}^{\infty}A_i)=\sum_{i=1}^{\infty}\mu(A_i)$.

Complete measure space. (X,Σ,μ)

If $S\in\Sigma$ and $\mu(S)=0$,

then $\mathcal{U}\subset S\Rightarrow\mathcal{U}\in\Sigma$.

$\in\Sigma$

$\mathbb{R}^2$,

$\subset\mathbb{R}^2$.

non-measurable set in $\mathbb{R}$. $\subset\mathbb{R}$.

Σ = generated by measurable $\times$ measurable.