Tutorial 4 Problems

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Recollections.

Last week you learned

- Structure theorem of open sets in $\mathbf{R}$. Think about the generalization in $\mathbf{R}^n$.
- The "volume" of open sets in $\mathbf{R}$. The volume of compact sets and closed sets in $\mathbf{R}$.
- Definition of outer & inner measures. The first definition for measurability: $E \subseteq \mathbf{R}$ is measurable iff. $\mu_*(E) = \mu^*(E)$.
- Properties about outer measure, in particular, the countably sub-additivity.
- The Cantor set, which will be summarized below.
- The Caratheodory condition, the second definition of measurability. It seems strange and less intuitive, but it is described in terms of outer measures only, and is not essentially related to the topology of $\mathbf{R}$, so it can be generalized to an abstract framework, as was already shown in the lecture on this Wednesday.

Reminder. In the logical order of learning, one has to learn some basic set theory and topology before learning measure theory, even the Lebesgue measure on $\mathbf{R}$. But the course is designed to avoid set-theoretic issues and directly learn measure theory. This might result in difficulty in understanding constructions that are in set-theoretic nature. If you want to get a

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complete understanding to this subject, I recommend you to learn some set theory as well as topology by yourself.

1 Warm-up.

$E \subseteq \mathbf{R}$, $\mu^*(E) > 0$. For any $m \in [0, \mu^*(E)]$, show that there exists $F \subseteq E$ s.t. $\mu^*(F) = m$.

2 Review of the Cantor Set

Let $C_0 = [0, 1]$, $C_1 = [0, \frac{1}{3}] \sqcup [\frac{2}{3}, 1]$, $\cdots$ be the sequence defined recursively by removing the middle $\frac{1}{3}$ of each piece of closed intervals of the last set. Let $C = \cap_{i=0}^{\infty} C_i$ be the Cantor set. Show that:

(i) C is non-empty.

(ii) C is compact.

(iii) C is measurable with measure zero. *That is to say, C is "small" from the measure-theoretic point of view.*

(iv) C is totally disconnected i.e. $\forall c_1, c_2 \in C, \exists c \notin C$ s.t. $c_1 < c < c_2$. This implies that C is **nowhere dense** i.e. C does not contain any non-empty open sets. *That is to say, C is "small" from the Baire category theoretic point of view.*

(v) C has no isolated points. That is to say, for any $c \in C$ and open neighborhood U of c, $U \cap C$ is not the single point set $\{c\}$.

(vi) C is not countable. Actually there exists a surjection $\phi : C \to [0, 1]$. *That is to say, C is "big" from the set-theoretic point of view.*

3 The Illusion of Measures

The early dream of "volume". At the beginning, mathematicians wanted to generalize the notion of "area" or "volume" in geometry. Concretely in the 3-dimensional case, they wanted to create a function

$$v : \text{subsets of } \mathbf{R}^3 \to [0, +\infty]$$

called the volume function, such that v is countably additive, invariant under rigid motions, and takes value 1 for the unit cube. This is an ideal definition for a volume function that is used to do geometry. But their dream turns out to be impossible, even in the 1-dimensional case:

A strange set. Define the following equivalence relation on $(0, 1)$: $x \sim y$ iff. $x - y \in \mathbf{Q}$. Let S be a set of representatives of the equivalence classes. Let $W = (-1, 1) \cap \mathbf{Q}$, for any $r \in W$, let $S_r = r + S = \{r + s : s \in S\}$. Show that:

(i) For any real number $x \in \mathbf{R}$, there exists a unique $s \in S$ s.t. $x - s \in \mathbf{Q}$.

(ii) For any $r \in W$, $S_r \subseteq (-1, 2)$.

(iii) For $r_1 \neq r_2 \in W$, $S_{r_1} \cap S_{r_2} = \phi$.

(iv) We have

$$(0, 1) \subseteq \bigsqcup_{r \in W} S_r \subseteq (-1, 2)$$

Now suppose we have a volume function v satisfying the above properties. Consider $v(S)$. Because of translation invariance (translation is a rigid motion), $v(S) = v(S_r)$ for any $r \in W$. Since W is countable, by countable additivity we have

$$1 \leq \sum_{w \in W} v(S) \leq 3$$

which is a contradiction. (If $v(S) > 0$, $\sum_{w \in W} v(S) = +\infty$; if $v(S) = 0$, $\sum_{w \in W} v(S) = 0$. Neither is possible because of the inequality above.)

Measurable sets. The above example shows that the desired volume function does not exist. So mathematicians just "give up" some awful sets like the S above, and developed Lebesgue measure theory for Lebesgue measurable sets.

Remark 3.0.1.

- *In the construction of the set S above, we have to choose a representative in each equivalence class, where we have to invoke* ***the axiom of choice****.*

- *If one wants a function that is only* ***finitely additive*** *rather than countably additive, unfortunately the answer is still negative. An interesting aspect is the following theorem, which is counter-intuitive:*

(Banach-Tarski) For arbitraty bounded open sets $U, V \subseteq \mathbf{R}^n (n \geq 3)$ there exists $k \in \mathbf{N}$ and disjoint partitions $U = \sqcup_{i=1}^k U_i, V = \sqcup_{i=1}^k V_i$ such that for each i, U_i can be transformed into V_i via rigid motions.