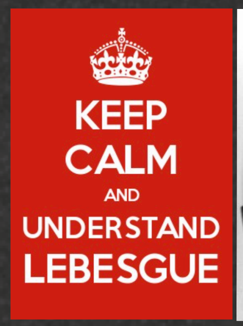
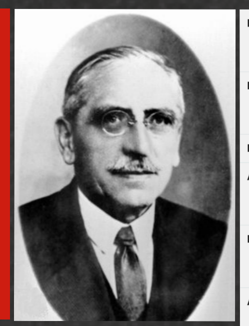
List of important topics for Ch.6 and 8:

- metric space, normed vector space
- open/closed/compact/etc.
- Bolzano-Weierstrass, Heine Borel, etc.
- Continuity, differentiability, C^k functions $F: \mathbb{R}^m \to \mathbb{R}^n$
- Jacobian, chain rule
- Inverse/Implicit Function Theorem → proof → uses ← especially geometric uses
- Higher order differentiability, $f_{xy} = f_{yx}$.





Born	June 28, 1875 Beauvais, Oise, France
Died	July 26, 1941 (aged 66) Paris, France
Nationality	French
Alma mater	École Normale Supérieure, University of Paris
Known for	Lebesgue integration, Lebesgue measure
Awards	Fellow of the Royal Society, Poncelet Prize for 1914

Scientific career

Fields	Mathematics
Institutions	University of Rennes, University of Poitiers, University of Paris, Collège de France
Doctoral advisor	Émile Borel
Doctoral students	Paul Montel, Zygmunt Janiszewski, Georges de Rham

Prop: Any open set in $\mathbb{R}$ is a countable disjoint union of open intervals.

Proof: $(x,a)\subset U$

$A_x := \sup\{a : (x,a)\subset U\}$

$B_x := \inf\{b : (b,x)\subset U\}$

$U = \coprod_{i=1}^{\infty}(a_i,b_i)$

$\lambda(U) := \sum_{i=1}^{\infty}(b_i-a_i)$

Claim: $U = \bigcup_{x\in U}(B_x, A_x)$

Disjoint? ← (B_x,A_x), (B_y,A_y) are either $=$ or disjoint.

if NOT: $\subset U$

Countable? Know: $U = \coprod_{\alpha\in A}(B_\alpha, A_\alpha)$.

$\forall \alpha\in A$ $\exists q_\alpha \in (B_\alpha,A_\alpha)\cap\mathbb{Q}$.

$A \to \mathbb{Q}$ (1-1), $\alpha \mapsto q_\alpha$.

$\mathbb{Q}$ countable ⇒ A too.

① $U \subset \bigcup_{i=1}^{\infty} V_i \Rightarrow \lambda(U) \le \sum_{i=1}^{\infty}\lambda(V_i)$

open in $\mathbb{R}$; open intervals; length

② $\lambda(U\cup V) = \lambda(U) + \lambda(V) - \lambda(U\cap V)$

open; open; open

Suppose ① and ② hold for U, V, V_i = FINITE union of disjoint open intervals. and $\bigcup_{i=1}^{n} V_i$

Suppose $\lambda(U)<\infty$, $\sum_{i=1}^{\infty}(b_i-a_i)<\infty$

① let $U = \coprod_{i=1}^{\infty}(a_i,b_i) \subset \bigcup_{j=1}^{\infty}(A_j,B_j)$

$U_n := \coprod_{i=1}^{n}(a_i,b_i)$ WANT: $\lambda(U) \le \sum_{j=1}^{\infty}(B_j-A_j)$.

$\lambda(U-U_n) = \lambda\left(\coprod_{i>n}(a_i,b_i)\right) = \sum_{i>n}(b_i-a_i)$

$K := [a_1+\varepsilon, b_1-\varepsilon]\sqcup\cdots\sqcup[a_n+\varepsilon, b_n-\varepsilon]$

compact

$K\subset U\subset \bigcup_{j=1}^{\infty}(A_j,B_j)$ ⇒ $\exists (A_{j_1},B_{j_1}),\dots,(A_{j_N},B_{j_N})$ s.t. $K\subset\bigcup_{k=1}^{N}(A_{j_k},B_{j_k})$ finite

$\lambda(K) = \sum_{i=1}^{n}(b_i-a_i) - 2n\varepsilon \le \sum_{k=1}^{N}(B_{j_k}-A_{j_k}) \le \sum_{j=1}^{\infty}(B_j-A_j)$

finite

$\forall n, \exists \varepsilon_0$ s.t. $\varepsilon<\varepsilon_0 \Rightarrow \sum_{i=1}^{n}(b_i-a_i) - 2n\varepsilon \le \sum_{j=1}^{\infty}(B_j-A_j)$

$\downarrow \varepsilon\to 0$, n fixed

$\forall n$: $\sum_{i=1}^{n}(b_i-a_i) \le \sum_{j=1}^{\infty}(B_j-A_j)$

$\downarrow n\to\infty$

$\lambda(U) = \sum_{i=1}^{\infty}(b_i-a_i) \le \sum_{j=1}^{\infty}(B_j-A_j)$.

② $U = \coprod_{i=1}^{\infty}(a_i,b_i)$, $V = \coprod_{j=1}^{\infty}(c_j,d_j)$.

$U_n := \coprod_{i=1}^{n}(a_i,b_i)$, $V_n := \coprod_{j=1}^{n}(c_j,d_j)$.

$\lambda(U-U_n)$ $\lambda(V-V_n)$

(*) $\boxed{\lambda(U_n\cup V_n) = \lambda(U_n)+\lambda(V_n) - \lambda(U_n\cap V_n)}$

$\lambda(U\cup V) - \lambda(U_n\cup V_n) <$?

$\lambda(U\cap V) - \lambda(U_n\cap V_n) <$?

$U\cup V = (U_n\cup V_n)\cup(U-U_n)\cup(V-V_n)$

$U\cap V \subset (U_n\cap V_n)\cup((U-U_n)\cup(V-V_n))$?

Exercise. $\sum_{i>n}(b_i-a_i)$

$0 \le \lambda(U\cup V) - \lambda(U_n\cup V_n) \le \lambda(U-U_n)+\lambda(V-V_n) \to 0$

$0 \le \lambda(U\cap V) - \lambda(U_n\cap V_n) \le \lambda(U-U_n)+\lambda(V-V_n) \to 0$.

■ by $\lim_{n\to\infty}$ (*).

$\lambda(K) := \lambda(U) - \lambda(U-K)$

compact; open; open

U open, $U-K$ open

U, V open, $\supset K$

$\lambda(U) - \lambda(U-K) = \lambda\big((U-K)\cup(U\cap V)\big) - \lambda(U-K)$

$\lambda\big((U-K)\cup(U\cap V)\big) = \lambda(U-K) + \lambda(U\cap V) - \lambda\big((U-K)\cap(U\cap V)\big)$

open; open

$\Rightarrow \lambda(U) - \lambda(U-K) = \lambda(U\cap V) - \lambda\big((U-K)\cap(U\cap V)\big)$

$= \underbrace{\lambda(U\cap V) - \lambda\big((U\cap V) - K\big)}_{\text{symmetric about } (U,V)}$.

$\therefore \lambda(U) - \lambda(U-K) = \lambda(V) - \lambda(V-K)$.

$A\subset U$, open set in $\mathbb{R}$, $\lambda(U)$

"length" of A

Lebesgue outer measure of A. U open, $\subset\mathbb{R}$

$\mu^*(A) := \inf\{\lambda(U) : A\subset U \text{ open}\}$

$\mu_*(A) = \sup\{\lambda(K) : K\subset A, K \text{ closed}\}$

Lebesgue inner measure of A.

closed, $\lambda(K)$, "length" of A

$A\subset\mathbb{R}$ is Lebesgue measurable $\overset{\text{def}}{\iff}$ $\mu^*(A) = \mu_*(A)$

$(\mu(A) := \mu^*(A))$.

$\mu^*((a,b)) = \lambda(a,b)$. $(a,b)\subset \underbrace{(a,b)}_{\text{open}}$.

$\mu^*((a,b))$ $[a+\varepsilon, b-\varepsilon]\subset(a,b)$

$\lambda([\ ])$ $\mu_*((a,b))$ $\lambda([a+\varepsilon, b-\varepsilon]) = b-a-2\varepsilon$

$b-a-2\varepsilon \le \mu_*((a,b))$ $\forall\varepsilon<\varepsilon_0$.

$\le \mu^*((a,b))$

$= b-a$ $\forall\varepsilon$ small.

$\varepsilon\to 0^+$: $\mu_*((a,b)) = \mu^*((a,b)) = b-a$.

Last time:

- U open in $\mathbb{R}$ ⇒ $U = \coprod_{i=1}^{\infty}(a_i,b_i)$, $\lambda(U) := \sum_{i=1}^{\infty}(b_i-a_i)$
- $\lambda\left(\bigcup_{i=1}^{\infty}U_i\right) \le \sum_{i=1}^{\infty}\lambda(U_i)$ (U_i open) If $U\cap V=\emptyset$ ⇒ $\lambda(U\cup V) = \lambda(U)+\lambda(V)$
- $\lambda(U\cup V) = \lambda(U) + \lambda(V) - \lambda(U\cap V)$ (open)
- K compact, $\lambda(K) := \lambda(U) - \lambda(U-K)$ ($U \supset K$ open)

  C closed. $\lambda(C) := \lim_{M\to+\infty}\lambda(C\cap[-M,M])$.
- $\mu^*(A) := \inf\{\lambda(U) : U\supset A, U \text{ open}\}$ Lebesgue outer measure

  $\mu_*(A) := \sup\{\lambda(K) : K\subset A, K \text{ closed}\}$. " inner "

$\lambda(K)$ μ_* μ^* $\lambda(U)$

$A\subset\mathbb{R}$ is Lebesgue measurable $\overset{\text{def}}{\iff}$ $\mu^*(A) = \mu_*(A)$

then $\mu(A) := \mu^*(A)$ Lebesgue measure

- Shown: $\mu((a,b)) = b-a$

- **Claim:** $\mu(\text{open set}) = \lambda(\text{open set})$

Proof: let $U = \coprod_{i=1}^{\infty}(a_i,b_i)$

$\forall n$. $K_n := [a_1+\varepsilon, b_1-\varepsilon]\sqcup\cdots\sqcup[a_n+\varepsilon, b_n-\varepsilon]$

$\lambda(K_n) \le \mu_*(U) \le \mu^*(U) = \lambda(U) = \sum_{i=1}^{\infty}(b_i-a_i)$

$K_n \subset U$; $\lambda(K_n) = \sum_{i=1}^{n}(b_i-a_i) - 2n\varepsilon$; def of μ^*

$\forall n, \forall\varepsilon$ small ⇒ $\sum_{i=1}^{n}(b_i-a_i) - 2n\varepsilon \le \mu_*(U) \le \mu^*(U) = \lambda(U) = \sum_{i=1}^{\infty}(b_i-a_i)$

$\downarrow \varepsilon\to 0$, n fixed

$\sum_{i=1}^{n}(b_i-a_i) \le \mu_*(U) \le \mu^*(U) =$

$\downarrow n\to\infty$

Claim: A is Lebesgue measurable

bounded

$\iff \forall\varepsilon>0, \exists K\subset A\subset U$ s.t. $\lambda(U-K)<\varepsilon$. (K closed, U open)

Proof: $(\Rightarrow)$ inf, inf $+\varepsilon/2$, $\lambda(K_\varepsilon)$, $\mu_* = \mu^*$

$\mu^*(A) = \inf\{\lambda(U) : U\supset A\}$

$\exists U_\varepsilon \supset A$ s.t.

$\mu^*(A) \le \lambda(U_\varepsilon) < \mu^*(A) + \varepsilon/2$

$\exists K_\varepsilon\subset A$ s.t.

$\mu_*(A) - \varepsilon/2 < \lambda(K_\varepsilon) \le \mu_*(A)$

$\lambda(K_\varepsilon) := \lambda(U_\varepsilon) - \lambda(U_\varepsilon - K_\varepsilon)$.

$\lambda(U_\varepsilon - K_\varepsilon) < \varepsilon$.

$(\Leftarrow)$ $\forall\varepsilon>0$, $\exists K_\varepsilon \subset A$, $U_\varepsilon \supset A$ s.t. $\lambda(U_\varepsilon - K_\varepsilon) < \varepsilon$.

$\lambda(K_\varepsilon)$ $\mu_*(A)$ $\mu^*(A)$ $\lambda(U_\varepsilon)$ $<\varepsilon$.

$\Rightarrow 0 \le \mu^*(A) - \mu_*(A) < \varepsilon$ $\forall\varepsilon>0$.

HW: A, B are Lebesgue measurable

$\Rightarrow A\cup B$, $A\cap B$, $A-B$ $(B-A)$ are all Lebesgue measurable.

$\mu(A\cup B) = \mu(A) + \mu(B) - \mu(A\cap B)$.

$A\cup B$: $\forall\varepsilon>0$, $\exists U_A\supset A\supset K_A$ s.t. $U_B\supset B\supset K_B$

$\begin{cases}\lambda(U_A-K_A)<\varepsilon/2\\ \lambda(U_B-K_A)<\varepsilon/2\end{cases}$

$\underbrace{U_A\cup U_B}_{\text{open}} \supset A\cup B\supset \underbrace{K_A\cup K_B}_{\text{closed}}$.

$X-Y = X\cap Y^c$, $Y^c = \mathbb{R}-Y$.

$\lambda((U_A\cup U_B) - (K_A\cup K_B))$

$= \lambda((U_A\cup U_B)\cap(K_A\cup K_B)^c)$

$= \lambda((U_A\cup U_B)\cap(K_A^c\cap K_B^c))$

$= \lambda\Big(\underbrace{(U_A\cap(K_A^c\cap K_B^c))}_{\subset U_A\cap K_A^c}\cup\underbrace{(U_B\cap(K_A^c\cap K_B^c))}_{\subset U_B\cap K_B^c}\Big)$

$\le \lambda((U_A-K_A)\cup(U_B-K_B))$

$\le \lambda(U_A-K_A) + \lambda(U_B-K_B) < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$.

$C \subset U$, C closed, bdd

$C = U-(U-C)$, $U-C$ open

$\lambda(C) + \lambda(\underbrace{U-C}_{\text{open}}) = \lambda(U)$

$\lambda(C) + \mu(U-C) = \mu(U)$

$= \mu((U-C)\sqcup C)$

$= \mu(U-C) + \mu(C)$.

$\Rightarrow \lambda(C) = \mu(C)$.

μ^* = Leb. outer meas. on $\mathbb{R}$.

Prop: ① $0 \le \mu_*(A) \le \mu^*(A)$

$K\subset A\subset U$, $\lambda(K)\le\lambda(U)$, $\lambda(K) = \lambda(U) - \lambda(U-K)$.

② $A\subset B$ $\Rightarrow \begin{cases}\mu^*(A)\le\mu^*(B)\\ \mu_*(A)\le\mu_*(B).\end{cases}$

$\{\lambda(U) : U\supset A\} \supset \{\lambda(U) : U\supset B\}$.

$\underbrace{\inf\{\cdots A\}}_{\mu^*(A)} \le \underbrace{\inf\{\cdots B\}}_{\mu^*(B)}$ $U\supset B \Rightarrow U\supset A$

③ $\mu^*\left(\bigcup_{i=1}^{\infty}A_i\right) \le \sum_{i=1}^{\infty}\mu^*(A_i)$ [true for all A_i's $\subset\mathbb{R}$]

(countable subadditivity)

Proof: $\lambda(U_i)$, $\mu^* = \inf$, $\mu^* + \varepsilon/2^i$ A_i, U_i

$\forall i$: $\mu^*(A_i) \le \lambda(U_i) < \mu^*(A_i) + \frac{\varepsilon}{2^i}$

proved last lect.

$\mu^*\left(\bigcup_{i=1}^{\infty}A_i\right) \le \lambda\left(\bigcup_{i=1}^{\infty}U_i\right) \le \sum_{i=1}^{\infty}\lambda(U_i)$

$\bigcup_{i=1}^{\infty}A_i \subset \bigcup_{i=1}^{\infty}U_i$ $\le \sum_{i=1}^{\infty}\mu^*(A_i) + \sum_{i=1}^{\infty}\frac{\varepsilon}{2^i}$ $= \varepsilon$.

$\downarrow \varepsilon\to 0$

$\mu^*\left(\bigcup_{i=1}^{\infty}A_i\right) \le \sum_{i=1}^{\infty}\mu^*(A_i)$

C = Cantor set.

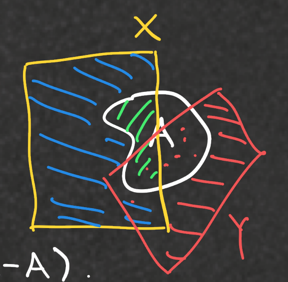
Carathéodory

Theorem: $A\subset\mathbb{R}$ bounded.

A is Lebesgue measurable

$\iff \forall X\subset\mathbb{R}$ bounded, we have

$\mu^*(X) = \mu^*(X\cap A) + \mu^*(X-A)$.



Lemma: $A, B\subset\mathbb{R}$ bounded. $A\cap B = \emptyset$.

$\mu_*(A\sqcup B) \le \mu_*(A) + \mu^*(B) \le \mu^*(A\sqcup B)$

Proof: GOAL $\mu_*(A) + \mu^*(B) < \mu^*(A\sqcup B) + \varepsilon$ $\forall\varepsilon>0$.

$\forall\varepsilon>0$. $\exists K\subset A$, $U\supset A\sqcup B$ s.t.

$\mu_*(A) - \frac{\varepsilon}{2} < \lambda(K) \le \mu_*(A)$

$\mu^*(A\sqcup B) \le \lambda(U) < \mu^*(A\sqcup B) + \varepsilon/2$

$\mu^*(B) \le \lambda(U-K) = \lambda(U) - \lambda(K) < \mu^*(A\sqcup B) + \varepsilon/2 - \mu_*(A) + \varepsilon/2$

$B \subset U-K$ open