Tutorial 3 Problems

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Recollections.

Last week you learned

- The implicit function theorem. Think of the geometric intuition.
- The definition of higher differentiability.
- Mixed partial derivatives. For a function $f : \mathbf{R}^2 \to \mathbf{R}$, Prof. FONG proved in his lecture that if f_x, f_y exists near $(0,0)$, differentiable at $(0,0)$, then $f_{xy}(0,0) = f_{yx}(0,0)$.

1 Warm-up.

Heat (diffusion) Equation. Verify that the function

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}}, t > 0, x \in \mathbf{R}$$

satisfies the Heat equation

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Harmonic Function. The **Laplace operator** (in Euclidean spaces) is $\Delta : f \mapsto \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$ for $f \in C^2(\mathbf{R}^n)$. A function in $C^2(\mathbf{R}^n)$ is called **harmonic** if $\Delta f = 0$. Verify the following functions are harmonic:

- $f(x,y) = \ln(x^2 + y^2)$

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- $f(x, y) = e^x \cos y$
- $f(x, y) = x^2 - y^2$

Judge the following statement. Let $U \subseteq \mathbf{R}^n$ be an open set in $\mathbf{R}^2$, $f \in C^1(U)$, $f_x = f_y = 0$ in U, then f is constant on U.

2 Mixed Partial Derivatives.

A Counter-example. Let

$$f(x, y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Prove that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

A Theorem. For a function $f : \mathbf{R}^2 \to \mathbf{R}$, if f_x, f_y exists near $(0, 0)$, f_{xy} exists near $(0, 0)$ and is continuous at $(0, 0)$, then $f_{yx}(0, 0)$ exists and $f_{xy}(0, 0) = f_{yx}(0, 0)$.

3 Compactness Revisited.

$f, g : \mathbf{R}^2 \to \mathbf{R}$ are C^1, such that $f_x g_y - f_y g_x \neq 0$ on $\mathbf{R}^2$. Show that the equation $f(x, y) = g(x, y) = 0$ only have finitely many solutions in any bounded closed set $E \subseteq \mathbf{R}^2$.

4 The Boss.

Let $D \subseteq \mathbf{R}^2$ be a convex open set in $\mathbf{R}^2$ containing $(0, 0)$, $f \in C^1(D)$ satisfies $xf_x + yf_y = 0$ in D. Show that $f(x, y)$ is constant in D.

Remark 4.0.1. *Think about what role does the **convexness** of D play in this problem? What if we drop this condition?*

The hint is on the next page.

(Hint: first prove that f is constant along each ray starting from zero. Then consider the behavior around zero.)