MATH 3043 • Fall 2019 • Honors Analysis II
Problem Set #1 • Due Date: 29/09/2019, 11:59PM

Instructions: Similar rules as in my former honor courses. Submit a clearly written or LaTeX-typed PDF to Canvas before the due time. Combine everything into one PDF in an organized manner. You can update your file as many times as you wish before the due time. **For each homework, a bonus point of 0.2 will be added to your course total score if it is fully typed by LaTeX (please upload BOTH the .tex and .pdf files).**

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1. (10 points) Let K be a non-empty compact subset of $\mathbb{R}^n$, and $\{K_\alpha\}_{\alpha\in\mathcal{A}}$ be an infinite (possibly uncountable) family of non-empty compact subsets of K. Suppose that for any finitely many $\{K_{\alpha_1}, \cdots, K_{\alpha_N}\}$ in this family, we must have $\cap_{i=1}^N K_{\alpha_i} \neq \emptyset$. Show that the intersection of all sets in the family is non-empty, i.e.

$$\bigcap_{\alpha\in\mathcal{A}} K_\alpha \neq \emptyset.$$

[Hint: proof by contradiction]

2. (20 points) Denote by $\mathcal{K}$ the set of all non-empty **compact** subsets of $\mathbb{R}^n$. For each $A, B \in \mathcal{K}$, we define

$$d(A,B) := \inf\{|x-y| : x \in A \text{ and } y \in B\},$$
$$d_H(A,B) := \max\left\{\sup_{x\in A}\inf_{y\in B}|x-y|\,,\ \sup_{y\in B}\inf_{x\in A}|x-y|\right\}.$$

   (a) Show that $d(A,B) = 0$ if and only if $A \cap B \neq \emptyset$.

   (b) Show that $d_H(A,B) = 0$ if and only if $A = B$.

   (c) Verify that $(\mathcal{K}, d_H)$ is a metric space. [FYI: d_H is called the Hausdorff metric.]

   (d) Fix $n = 2$, i.e. consider $\mathbb{R}^2$. For each $a, b > 0$, denote by $E(a,b)$ the ellipse in $\mathbb{R}^2$ given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It is obvious (to MATH 3043 students) that $E(a,b) \in \mathcal{K}$. Now given a sequence $\{(a_i,b_i)\}_{i=1}^\infty$ with $a_i, b_i > 0$. Show that $\{(a_i,b_i)\}_{i=1}^\infty$ is a Cauchy sequence with respect to the L_2-norm if and only if $\{E(a_i,b_i)\}_{i=1}^\infty$ is a Cauchy sequence with respect to the d_H-metric.

3. (10 points) Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

Show that:

   (a) f is C^1 on $\mathbb{R}^2$.

   (b) $\dfrac{\partial}{\partial y}\dfrac{\partial f}{\partial x} \neq \dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial y}$ at $(0,0)$.

4. (15 points) Let $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^m \to \mathbb{R}^k$ be two C^1 functions.

   (a) Complete the proof of the chain rule $D(G \circ F)(x_0) = DG(F(x_0))DF(x_0)$ as outlined in class.

   (b) Show that if both F and G are C^k, then so is $G \circ F$.

5. (10 points) Consider a map $F : \overline{B_R(x_0)} \to \mathbb{R}^n$ of the form $F = \mathrm{id} + G$ where id is the identity map, and $G : \overline{B_R(x_0)} \to \mathbb{R}^n$ is a map such that there exists $\alpha \in (0,1)$ so that

$$|G(x_1) - G(x_2)| \leq \alpha\, |x_1 - x_2|$$

for $x_1, x_2 \in \overline{B_R(x_0)}$. Suppose $F(x_0) = y_0$. Show that there exist $\varepsilon, r > 0$ such that for any $y \in B_\varepsilon(y_0)$, there exists a unique $x \in \overline{B_r(x_0)}$ such that $F(x) = y$.

[Hint: Modify the proof of inverse function theorem.]

6. (10 points) Given a C^1 function $f : \mathbb{R}^{k+n} \to \mathbb{R}^n$ satisfying $f(x_0) = 0$ for some $x_0 \in \mathbb{R}^{k+n}$, and $Df(x_0)$ has rank n. Show that for exists $\varepsilon > 0$ such that for any $y \in B_\varepsilon(0)$, the equation $f(x) = y$ has a solution.

7. (10 points) Let $\Sigma \subset \mathbb{R}^3$ be a surface which can be parametrized by two different bijective maps:

$$\begin{aligned} F(u_1, u_2) &= \big(f_1(u_1,u_2), f_2(u_1,u_2), f_3(u_1,u_2)\big) : U \to \Sigma \\ G(u_1, u_2) &= \big(g_1(v_1,v_2), g_2(v_1,v_2), g_3(v_1,v_2)\big) : U \to \Sigma \end{aligned}$$

where the common domain $U \subset \mathbb{R}^2$ is a non-empty open set. Both F and G are C^1. Suppose further that

$$\frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2} \neq 0 \quad \text{and} \quad \frac{\partial G}{\partial v_1} \times \frac{\partial G}{\partial v_2} \neq 0 \quad \text{on } U.$$

Show that $G^{-1} \circ F$ is C^1.

8. (15 points) Recall that the Taylor's theorem with integral remainder is given by:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \int_{x_0}^{x} \frac{f^{(k+1)}(t)}{k!}(x - t)^k \, dt.$$

for any C^{k+1} function $f : \mathbb{R} \to \mathbb{R}$.

Now given a multivariable C^{k+1} function $g : \mathbb{R}^n \to \mathbb{R}$. Using the above result, prove the Taylor's theorem with integral remainder for multivariable functions: for any $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{a} = (a_1, \cdots, a_n)$, we have

$$\begin{aligned} g(\mathbf{x}) = g(\mathbf{a}) &+ \sum_{j=1}^{k} \sum_{\beta_1 + \cdots + \beta_n = j} \frac{1}{\beta_1! \cdots \beta_n!} \frac{\partial^j g}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}(\mathbf{a}) \cdot (x_1 - a_1)^{\beta_1} \cdots (x_n - a_n)^{\beta_n} \\ &+ \sum_{\beta_1 + \cdots + \beta_n = k+1} \frac{k+1}{\beta_1! \cdots \beta_n!}(x_1 - a_1)^{\beta_1} \cdots (x_n - a_n)^{\beta_n} \int_0^1 (1-t)^k \left.\frac{\partial^{k+1} g}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}\right|_{\mathbf{a} + t(\mathbf{x} - \mathbf{a})} dt \end{aligned}$$

where $\beta_1, \cdots, \beta_n$ are non-negative integers.

Remark: To simplify the notations, for each $\beta = (\beta_1, \cdots, \beta_n)$ such that $\beta_1 + \cdots + \beta_n = j$, you may denote

$$\begin{aligned} |\beta| &:= j \\ D^\beta g &:= \frac{\partial^j g}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} \\ (\mathbf{x} - \mathbf{a})^\beta &= (x_1 - a_1)^{\beta_1} \cdots (x_n - a_n)^{\beta_n} \end{aligned}$$

Last update: 16:15, 21/09/2019