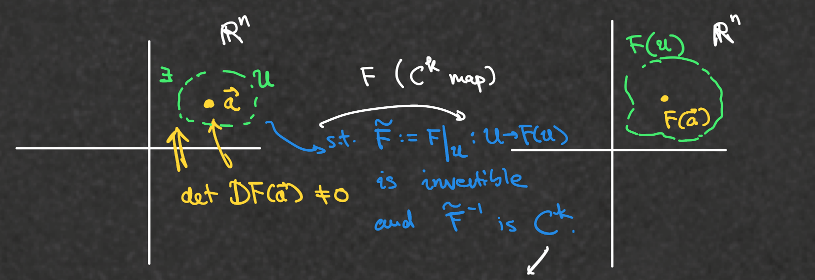



$\det DF(\vec{a}) \neq 0$

s.t. $\tilde{F} := F|_{\mathcal{U}} : \mathcal{U} \to F(\mathcal{U})$ is invertible and $\tilde{F}^{-1}$ is $C^k$.

$D(\tilde{F}^{-1}) = (DF)^{-1}$

$C^\infty$.

$F(\rho, \theta, \varphi) = (x, y, z) : (0,\infty)\times(-\infty,\infty)\times(0,\pi) \longrightarrow \mathbb{R}^3 \setminus z\text{-axis}.$

$x = \rho \sin\varphi \cos\theta$ $\qquad \det DF = \rho^2 \sin\phi.$

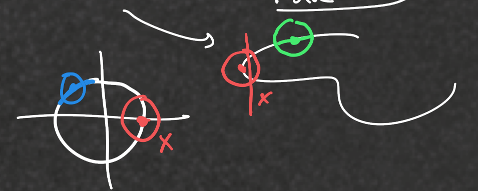
$y = \rho \sin\varphi \sin\theta$ $\qquad \hookrightarrow \tilde{F}^{-1}$ is $C^\infty$.

$z = \rho \cos\varphi.$

Implicit Function Theorem

$y^3 + 2xy + x^2 = 1$, find $\boxed{\frac{dy}{dx} = ?}$

$x^2 + y^2 = 1$

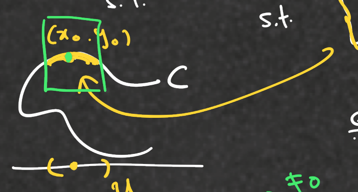


$F: \mathbb{R}^2 \to \mathbb{R}$, $C^1$. Consider $C = \{(x,y) : F(x,y) = 0\}$.

Suppose $\exists (x_0, y_0) \in C$ s.t. $\left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0)} \neq 0$

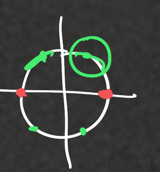
then $\exists\ \underset{x_0}{\mathcal{U}} \overset{\text{open}}{\subset} \mathbb{R}$, $\exists\ C^1$ $g: \mathcal{U} \to \mathbb{R}$, and $\underset{(x_0,y_0)}{V} \subset \mathbb{R}^2$

s.t. $\{(x, g(x)) : x \in \mathcal{U}\} = C \cap V$.

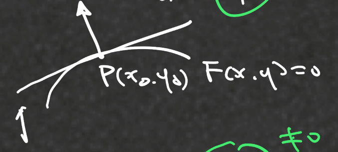



eg. $\underbrace{x^2 + y^2 - 1}_{F(x,y)} = 0$

$\frac{\partial F}{\partial y} = 2y$, $\frac{\partial F}{\partial x} = 2x$



$\nabla F = \frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j}$ $\neq 0$



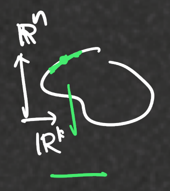

$\frac{\partial F}{\partial x}(P)\cdot(x - x_0) + \frac{\partial F}{\partial y}(P)\cdot(y - y_0) = 0.$ ($\neq 0$) $\rightsquigarrow y =$ in terms of $x$

$F: \underset{(\vec{x}, \vec{y})}{\mathbb{R}^{k+n}} \to \mathbb{R}^n$ $(k \geq 0)$, $C^1$, consider $\boxed{F^{-1}(\vec{c})}$

$(\vec{x}, \vec{y})$, $\vec{x} \in \mathbb{R}^k$, $\vec{y} \in \mathbr{R}^n$ Consider $\{(\vec{x}, \vec{y}) : F(\vec{x}, \vec{y}) = \vec{c}\}$ (const.)

$DF(\vec{x}_0, \vec{y}_0) = \Big[\ \boxed{A}\ \Big]_n$ with columns $\underbrace{\frac{\partial}{\partial x_i}\ \ \frac{\partial}{\partial y_j}}_{k+n}$

Suppose $A$ is invertible

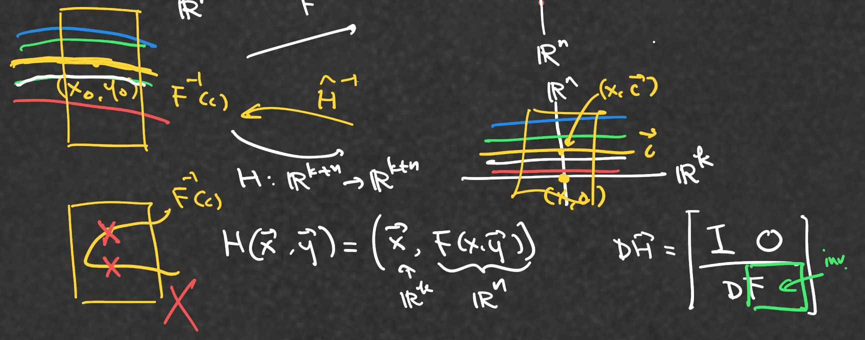



$\Rightarrow$ locally around $(x_0, y_0)$,

$F^{-1}(\vec{c})$ is a $C^1$ graph over $\vec{x} \in \mathbb{R}^k$.

$(\vec{y} = g(\vec{x}))$

Proof

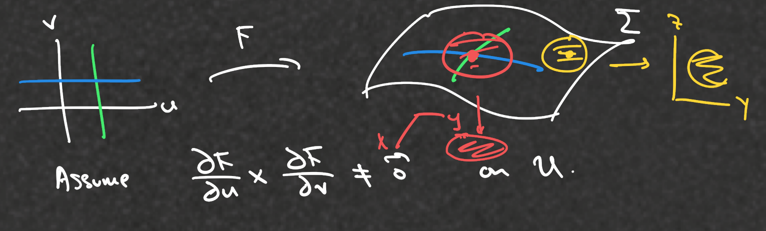



$H: \mathbb{R}^{k+n} \to \mathbb{R}^{k+n}$

$H(\vec{x}, \vec{y}) = (\underset{\mathbb{R}^k}{\vec{x}}, \underbrace{F(\vec{x}, \vec{y})}_{\mathbb{R}^n})$ $\qquad D\vec{H} = \begin{bmatrix} I & 0 \\ \multicolumn{2}{c}{DF} \end{bmatrix}$ inv.

$y = \pi_2 \circ \tilde{H}^{-1}(x, \vec{c})$ $\qquad \pi_2(\vec{x}, \vec{y}) = \vec{y}$.

$F(u,v) : \mathcal{U} \subset \mathbb{R}^2 \xrightarrow{\text{inj.}} \mathbb{R}^3$. $C^1$




Assume $\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} \neq \vec{0}$ on $\mathcal{U}$.

<u>Claim:</u> locally around $\forall P \in \Sigma$,

$\Sigma$ can be regarded as a $C^1$ graph over two of the variables $(x, y, z)$.

<u>Proof:</u> $F(u,v) = (x(u,v), y(u,v), z(u,v))$

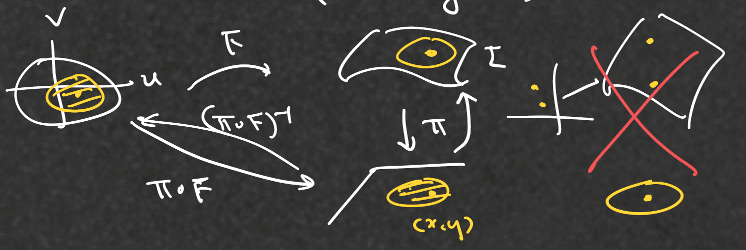
$0 \neq \frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$ $\begin{matrix} \leftarrow \frac{\partial F}{\partial u} \\ \leftarrow \frac{\partial F}{\partial v} \end{matrix}$.

WLOG

assume $\hat{k}$-component $\neq 0$ $\Longrightarrow$ $\det \frac{\partial(x,y)}{\partial(u,v)} \neq 0$.

$\pi(x, y, z) = (x, y)$

$\pi \circ F(u,v) = (x(u,v), y(u,v))$ $\qquad \det D(\pi \circ F) \neq 0$




$z(x,y) = \pi_3 \circ F\,(\pi \circ F)^{-1}(x,y)$ $\qquad \pi_3(x, y, z) = z$.

∎

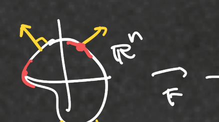
<u>Ex:</u>

$F: \mathbb{R}^n \to \mathbb{R}$, $C^1$. consider $F^{-1}(0)$.

Given: $\nabla F(\vec{x}) \neq \vec{0}$ $\quad \forall \vec{x} \in F^{-1}(0)$.

then locally around every $\vec{x} \in F^{-1}(0)$,

the hypersurface $F^{-1}(0)$ is a $C^1$ graph near $\vec{x}$.




<u>Proof:</u> $\nabla F = \left(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}\right) \neq \vec{0}$ at $\vec{a}$

$\Rightarrow \exists i$, $\frac{\partial F}{\partial x_i}(\vec{a}) \neq 0$.

$k+n$ $\quad \Big[\ \boxed{\diagup\diagup}\ \Big]_n$

$DF(\vec{a}) = \left[\frac{\partial F}{\partial x_1}\ \ \frac{\partial F}{\partial x_i} \cdot \frac{\partial F}{\partial x_n}\right]_{\vec{a}}$.

$\Downarrow$

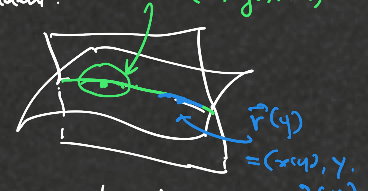
locally near $\vec{a}$. $\quad x_i = g(x_1, \dots, \hat{x}_i, \dots, x_n)$

<u>Ex:</u> $F^{-1}(0)$, $G^{-1}(0)$. , $F: \mathbb{R}^3 \to \mathbb{R}$, $G: \mathbb{R}^3 \to \mathbb{R}$.

$\{\nabla F, \nabla G\}$ linearly independent. $\qquad \vec{r}(x) = (x, y(x), z(x))$

$C = F^{-1}(0) \cap G^{-1}(0)$.

$\{F(x,y,z) = G(x,y,z) = 0\}$




<u>Claim:</u> $C$ can be locally parametrized by one of $x, y, z$.

<u>Proof:</u> $\vec{H}(x, y, z) : \mathbb{R}^3 \to \mathbb{R}^2$.

$= (F(x,y,z), G(x,y,z))$. $\qquad C = \vec{H}^{-1}\{(0,0)\}$.

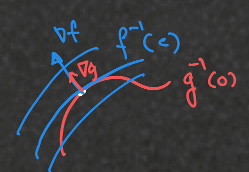
$D\vec{H} = \begin{bmatrix} -\nabla F- \\ -\nabla G- \end{bmatrix}$ > linearly indep. $\Rightarrow$ row rank $= 2$

$= \Big[\ \boxed{|\ |}\ \Big]$ $\Rightarrow$ col rank $= 2$.

linearly indep.

$\hookrightarrow$ $C$: $y, z =$ functions of $x$.

Similar if other two cols are linearly indep.

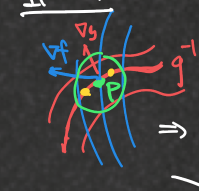



$g: \mathbb{R}^2 \to \mathbb{R}$

Optimize $f: \mathbb{R}^2 \to \mathbb{R}$ subject to $g^{-1}(0)$.

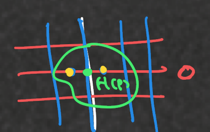
Solve $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases}$ Lagrange multiplier.

<u>If NOT:</u>




$\Rightarrow$ $\{\nabla f(P), \nabla g(P)\}$ linearly indep.

$H(x,y) = (f(x,y), g(x,y))$




$DH(P) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}_P = \begin{bmatrix} -\nabla f- \\ -\nabla g- \end{bmatrix}_P$ > linearly indep.

$\therefore$ invertible.

---

$f: \mathbb{R}^n \to \mathbb{R}$ is $k$th differentiable at $\vec{a}$

$\overset{\text{def}}{\iff}$ $\exists$ $k$-th degree polynomial $P_k(\vec{x})$

s.t. $f(\vec{x}) = P_k(\vec{x}) + o(\|\vec{x} - \vec{a}\|^k)$.

---

<u>Prop:</u> $f: \mathbb{R}^n \to \mathbb{R}$, given ① $\frac{\partial f}{\partial x_i}$ exist near $\vec{a}$ $\forall i$ on $B_\varepsilon(\vec{a})$

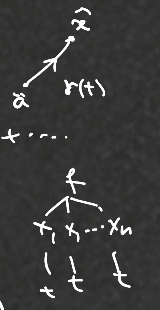
and they are ② differentiable at $\vec{a}$

$\Longrightarrow$ $f$ is twice differentiable at $\vec{a}$.

<u>Proof:</u> (Idea: How to guess $P_2(x)$?)

$\gamma(t) = \vec{a} + t(\vec{x} - \vec{a})$

$f(\gamma(t)) = f \circ \gamma(0) + \underbrace{(f \circ \gamma)'(0)} \cdot t + \underbrace{\frac{(f \circ \gamma)''(0)}{2!}} t^2 + \cdots$




$\frac{d}{dt}(f \circ \gamma) = \sum_i \frac{\partial f}{\partial x_i} \frac{d(a_i + t(x_i - a_i))}{dt} = \sum_i \left.\frac{\partial f}{\partial x_i}\right|_{\gamma(t)} \cdot (x_i - a_i)$.

$\frac{d^2}{dt^2}(f \circ \gamma) = \sum_{i,j=1}^n \left.\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)\right|_{\gamma(t)} (x_i - a_i)(x_j - a_j)$

$f \circ \gamma(1) = f(\vec{x})$

<u>Expect:</u> $P_2(x) = f(\vec{a}) + \sum_i \left.\frac{\partial f}{\partial x_i}\right|_{\vec{a}} \cdot (x_i - a_i)$

$+ \frac{1}{2!} \sum_{i,j=1}^n \left.\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)\right|_{\vec{a}} (x_i - a_i)(x_j - a_j)$

$f(\vec{x}) - P_2(\vec{x})$

$E_2(t) := \underbrace{(f \circ \gamma)}_{g}(t) - g(0) - \frac{g'(0)}{1!} t - \frac{g''(0)}{2!} t^2$ $\qquad$ $\vec{a}$ ($t=0$), $\vec{x}$ ($t=1$)

$E_2(1) = f(\vec{x}) - f(\vec{a}) - \frac{g'(0)}{1!} - \frac{g''(0)}{2!} = f(\vec{x}) - P_2(\vec{x})$

$\frac{E_2(1) - E_2(0)}{1^2 - 0^2} = \frac{E_2'(c)}{2c}$ $\qquad \frac{E_2(t)}{t^2}$

Cauchy MVT

$E_2(1) = \frac{E_2'(c)}{2c} = \frac{g'(c) - g'(0) - c\,g''(0)}{2c}$

$= \frac{1}{2c}\Big( \sum_{i=1}^n \left.\frac{\partial f}{\partial x_i}\right|_{\gamma(c)} \cdot (x_i - a_i) - \sum_{i=1}^n \left.\frac{\partial f}{\partial x_i}\right|_{\vec{a}} (x_i - a_i)$ $\qquad$ $\frac{\partial f}{\partial x_i}$ diff.

$- c \sum_{i,j=1}^n \left.\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)\right|_{\vec{a}} (x_i - a_i)(x_j - a_j) \Big)$ $\qquad$ $\gamma(c) - \vec{a} = \cancel{\vec{a}} + c(\vec{x} - \vec{a})$

$= \frac{1}{2\cancel{c}}\Big( \sum_{i=1}^n \Big( \cancel{\left.\frac{\partial f}{\partial x_i}\right|_{\vec{a}}} + \sum_{j=1}^n \left.\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)\right|_{\vec{a}} (\gamma_j(c) - a_j) + o(\|\gamma(c) - \vec{a}\|) \Big)(x_i - a_i)$ $\qquad$ $c(x_j - a_j)$, $o(\cancel{c}\|\vec{x} - \vec{a}\|)$

$- \cancel{\sum_{i=1}^n \left.\frac{\partial f}{\partial x_i}\right|_{\vec{a}} (x_i - a_i)} - \cancel{c \sum_{i,j=1}^n \left.\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)\right|_{\vec{a}} (x_i - a_i)(x_j - a_j)} \Big)$

$\leq \frac{1}{2} o(\|\vec{x} - \vec{a}\|) \cdot \underbrace{\sum_i |x_i - a_i|}_{\leq C\|\vec{x} - \vec{a}\|} \leq o(\|\vec{x} - \vec{a}\|^2)$

∎

$f(x,y)$ $\qquad \nabla f(a,b) = 0$

$= f(a,b) + \cancel{\frac{\partial f}{\partial x}(a,b)} \cdot (x-a) + \cancel{\frac{\partial f}{\partial y}(a,b)} \cdot (y-b)$

$+ \frac{1}{2!}\Big( \frac{\partial^2 f}{\partial x^2}(a,b) \cdot (x-a)^2 + \left.\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right|_{(a,b)} (x-a)(y-b)$

$+ \left.\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right|_{(a,b)} (x-a)(y-b) + \left.\frac{\partial^2 f}{\partial y^2}\right|_{(a,b)} (y-b)^2 \Big)$

$+ o(\|(x,y) - (a,b)\|^2)$.

$+$ or $-$ ? $\qquad$ eigenvalues $+$ or $-$ ?

$\frac{1}{2!} \begin{bmatrix} x-a & y-b \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}_{(a,b)} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$.

$f_{xx}(a,b) + f_{yy}(a,b) = \lambda_1 + \lambda_2$

$(f_{xx} f_{yy}^2 - f_{xy} f_{yx})(a,b) = \lambda_1 \lambda_2$

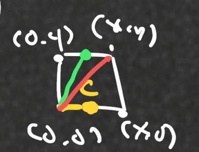
---

<u>Theorem:</u> $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ $\qquad (0,0)$.

Given $\begin{cases} f_x, f_y \text{ exist near } (0,0) \\ \text{they are differentiable at } (0,0). \end{cases}$

then $f_{xy}(0,0) = f_{yx}(0,0)$ $\qquad \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}$

<u>Proof:</u> $(\underbrace{f(x,y) - f(x,0)}_{=: g(x)}) - (\underbrace{f(0,y) - f(0,0)}_{g(0)})$ ——— $(*)$




$= g'(c) \cdot (x - 0)$

$= (f_x(c,y) - f_x(c,0)) \cdot x$

$= \begin{pmatrix} \cancel{f_x(0,0)} + \cancel{(f_x)_x(0,0)} \cdot c + (f_x)_y(0,0) \cdot y + o(\|(c,y)\|) \\ -\cancel{f_x(0,0)} - \cancel{(f_x)_x(0,0)} \cdot c - (f_x)_y(0,0) \cdot 0 + o(\|(c,0)\|) \end{pmatrix} x$ $\qquad = o(\|(x,y)\|)$

$= ((f_x)_y(0,0) \cdot y + o(\|(x,y)\|))\, x$ $\qquad \leq \|(x,y)\|$.

$= f_{xy}(0,0) \cdot xy + o(\|(x,y)\|^2)$

Similar, $(*) = (f(x,y) - f(0,y)) - (f(x,0) - f(0,0))$

$= f_{yx}(0,0)\, xy + o(\|(x,y)\|^2)$

$\Rightarrow (f_{xy}(0,0) - f_{yx}(0,0))\, xy = o(\|(x,y)\|^2)$

Along $\underline{x = y}$ $\quad \overset{x}{\cancel{y}} \to 0$

$(f_{xy}(0,0) - f_{yx}(0,0))\, \cancel{x^2} = o(\cancel{4x^2}) = o(1) \to 0$

$\therefore \underbrace{\phantom{xx}}\ = 0$.

∎