Tutorial 2 Problems

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Recollections.

Last week you learned

- Limit of multi-variable functions.
- Differentiability of multi-variable functions. Recall that

  (a) $f : \mathbf{R}^2 \to \mathbf{R}$ is differentiable at $(a, b) \Rightarrow f$ is continuous at (a,b) and the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ at (a, b) exists.

  (b) The partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist near (a, b) and are continuous $\Rightarrow f$ is differentiable at (a, b).

- The Jacobian of a multi-variable vector-valued function, the chain rule.
- The inverse function theorem.

  Theorem 0.0.1. *Suppose $F : \mathbf{R}^n \to \mathbf{R}^n$ is C^1 near 0, $F(0) = 0$, $(DF)(0) = I$. Then there exists $r > 0$ s.t. $F|_{B_r} : B_r \to F(B_r)$ is bijective, and the inverse map $F^{-1} : F(B_r) \to B_r$ is also C^1.*

  Remark 0.0.2. *It is cumbersome, yet possible, to define a function to be C^1 (or even continuous!) on a generic subset $A \subseteq \mathbf{R}^n$ which is NOT open. However in our setting above, it is possible to prove that $F(B_r)$ is OPEN. If we only suppose F to be continuous, this result is called **invariance of domains** whose original proof involves the Browner's fixed point theorem, or other input from algebraic topology which are far from trivial, such as the Jordan closed curve theorem. Is there a simple proof in our setting that f is C^1?*

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Recall the idea of the proof: Consider the map $T_y : \mathbf{R}^n \to \mathbf{R}^n, x \mapsto x - f(x) + y$. Try to prove that for r sufficiently small and $y \in B_r$, T_y maps B_r into B_r, and T_y is a contraction map. By the Banach's contraction mapping theorem, T_y admits a fixed point, namely there $\exists! x \in B_r$ s.t. $f(x) = y$, so we can costruct the inverse map. Then verify that the inverse map is C^1.

1 Warm-Up.

- Let $f : \mathbf{R} \to \mathbf{R}^3$ be a vector-valued function such that $||f(t)|| = 1$ for any $t \in \mathbf{R}$. Prove that $f'(t) \cdot f(t) = 0$ for any $t \in \mathbf{R}$. What is the geometric meaning of this result?

  Remark 1.0.1. *This gives an example of* ***immersion*** *of manifolds.*

- What can you say about $\lim_{(x,y)\to(+\infty,+\infty)}(\frac{xy}{x^2+y^2})^{x^2}$?

2 Various (counter-)examples.

- Let
$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$
  Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

- Let
$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$
  Show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

- Let
$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^3} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$
  Show that $F(r) = f(r\cos\theta, r\sin\theta)$ is continuous at $r = 0$ for any fixed $\theta \in [0, 2\pi]$, but f is not continuous at $(0,0)$.

- Let (again)

$$f(x,y)=\begin{cases}\frac{x^2y}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

  Show that f has partial derivatives at $(0,0)$, but f is not differentiable at $(0,0)$.

- Let

$$f(x,y)=\begin{cases}(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

  Show that f is differentiable at $(0,0)$, but the partial derivatives are not continuous at $(0,0)$.

3 About Change of the Order.

- Let

$$f(x,y)=\begin{cases}\frac{x^2-y^2}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

  Show that $\lim_{x\to 0}\lim_{y\to 0} f(x,y)\neq\lim_{y\to 0}\lim_{x\to 0} f(x,y)$.

- Let

$$f(x,y)=\begin{cases}(x^2+y^2)\sin\frac{1}{y} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

  Show that $\lim_{(x,y)\to(0,0)} f(x,y)=0$, but for $x\neq 0$, $\lim_{y\to 0} f(x,y)$ does not exists, so $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ does not make sense.

- Prove that if $\lim_{(x,y)\to(0,0)} f(x,y)=A$ exists, and for $x\neq 0$, $\lim_{y\to 0} f(x,y)$ exists, then $\lim_{x\to 0}\lim_{y\to 0} f(x,y)=A$.

4 The Boss.

If $f:\mathbf{R}^2\to\mathbf{R}$ satisfies:

- $f(x,y)$ is monotone in x for any $y\in\mathbf{R}$;
- $f(x,y)$ is conitnuous in x for any $y\in\mathbf{R}$, continuous in y for any $x\in\mathbf{R}$;

Prove that f is continuous.