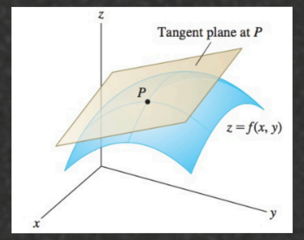
$f(x,y): \mathbb{R}^2 \to \mathbb{R}$ is said to be differentiable at (a,b)



$\overset{\text{def}}{\iff}$ $\exists$ a linear map $L: \mathbb{R}^2 \to \mathbb{R}$, $L(x,y) = Ax + By$ (A, B constants)

s.t. $f(x,y) = f(a,b) + L(x-a, y-b) + o(\|(x,y)-(a,b)\|)$ as $(x,y)\to(a,b)$

i.e. $\displaystyle\lim_{(x,y)\to(a,b)} \frac{f(x,y) - f(a,b) - L(x-a,y-b)}{\|(x,y)-(a,b)\|} = 0$

Prop: $f: \mathbb{R}^2 \to \mathbb{R}$ diff^{ble} at (a,b) $\Longrightarrow$ $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at (a,b).

Proof: $(x,y) = (a+t, b) \xrightarrow{t\to 0} (a,b)$, $(x,y)-(a,b) = (t,0)$

$f(a+t,b) = f(a,b) + A((a+t)-a) + B(b-b) + o(|t|)$ as $t\to 0$.

$f(a+t,b) - f(a,b) = At + o(|t|)$

$\left|\frac{1}{t}(f(a+t,b)-f(a,b)) - A\right| = o\left(\frac{|t|}{|t|}\right) = o(1)$.

$t\to 0$: $\left|\frac{\partial f}{\partial x}(a,b) - A\right| = 0 \Rightarrow A = \frac{\partial f}{\partial x}(a,b)$

Similarly $(a, b+t)$: $B = \frac{\partial f}{\partial y}(a,b)$.

Cor: f is diff. at $(a,b) \Rightarrow f(x,y) = \underbrace{f(a,b) - \frac{\partial f}{\partial x}(a,b)\cdot(x-a) - \frac{\partial f}{\partial y}(a,b)\cdot(y-a)}_{\text{tangent plane}} + o(\|(x,y)-(a,b)\|)$ as $(x,y)\to(a,b)$.

Example: $f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0). \end{cases}$

$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0}\frac{f(0+t,0) - \cancel{f(0,0)}}{t} = \lim_{t\to 0}\frac{0}{t} = 0.$

$\frac{\partial f}{\partial y}(0,0) = 0.$

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - 0\cdot(x-\cancel{x}) - 0\cdot(y-\cancel{y})}{\|(x,y)\|}$

$= \displaystyle\lim_{(x,y)\to(0,0)} \frac{\frac{x^2y}{x^2+y^2}}{\sqrt{x^2+y^2}} = \lim_{r\to 0^+}\frac{\frac{r^3\cos^2\theta\sin\theta}{r^2}}{r} = \lim_{r\to 0^+}\underbrace{\cos^2\theta\sin\theta}_{\text{depend only }\theta}.$

limit doesn't exist.

$f(x,y)$ is not differentiable at $(0,0)$.

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f is diff. at (a,b)

$\Rightarrow f(x,y) = f(a,b) - \frac{\partial f}{\partial x}(a,b)\cdot(x-a) - \frac{\partial f}{\partial y}(a,b)\cdot(y-b) + o(\|(x,y)-(a,b)\|)$

$\vec{u} = (u_1, u_2)$ unit. $(x,y) = (a,b) + t(u_1,u_2)$

$f((a,b)+t(u_1,u_2)) = f(a,b) - \frac{\partial f}{\partial x}(a,b)\cdot tu_1 - \frac{\partial f}{\partial y}(a,b)\cdot tu_2 + o(|t|)$ as $t\to 0$

$\left|\frac{f((a,b)+t(u_1,u_2)) - f(a,b)}{t} - \left(\frac{\partial f}{\partial x}(a,b)\cdot u_1 + \frac{\partial f}{\partial y}(a,b)\cdot u_2\right)\right| = o(1)$ as $t\to 0$.

$\displaystyle\lim_{t\to 0}\frac{f((a,b)+t(u_1,u_2)) - f(a,b)}{t} = \frac{\partial f}{\partial x}(a,b)\cdot u_1 + \frac{\partial f}{\partial y}(a,b)\cdot u_2$

$\Longrightarrow D_{(u_1,u_2)}f(a,b) = \nabla f(a,b)\cdot(u_1,u_2).$

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f is C^1 on $U \subset \mathbb{R}^2$ (open) $\overset{\text{def}}{\iff}$ $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist on U and are continuous on U.

Prop: f is C^1 on $U \Longrightarrow f$ is differentiable on U.

Proof: $f(x,y) - f(a,b) - \frac{\partial f}{\partial x}(a,b)\cdot(x-a) - \frac{\partial f}{\partial y}(a,b)\cdot(y-b)$

$= f(x,y) - f(a,y) + f(a,y) - f(a,b) - \frac{\partial f}{\partial x}(a,b)\cdot(x-a) - \frac{\partial f}{\partial y}(a,b)\cdot(y-b)$

$= \frac{\partial f}{\partial x}\Big|_P\cdot(x-a) + \frac{\partial f}{\partial y}\Big|_Q\cdot(y-b) - \frac{\partial f}{\partial x}\Big|_{(a,b)}\cdot(x-a) - \frac{\partial f}{\partial y}\Big|_{(a,b)}\cdot(y-b)$

$= \left(\frac{\partial f}{\partial x}\Big|_P - \frac{\partial f}{\partial x}\Big|_{(a,b)}\right)(x-a) + \left(\frac{\partial f}{\partial y}\Big|_Q - \frac{\partial f}{\partial y}\Big|_{(a,b)}\right)(y-b)$

$|y-b|, |x-a| \le \|(x,y)-(a,b)\|$

$\left|\frac{f(x,y) - f(a,b) - \frac{\partial f}{\partial x}(a,b)\cdot(x-a) - \frac{\partial f}{\partial y}(a,b)\cdot(y-b)}{\|(x,y)-(a,b)\|}\right|$

$\le \underbrace{\left|\frac{\partial f}{\partial x}\Big|_P - \frac{\partial f}{\partial x}\Big|_{(a,b)}\right|}_{\to 0,\ \frac{\partial f}{\partial x}\text{ continuous}} + \underbrace{\left|\frac{\partial f}{\partial y}\Big|_Q - \frac{\partial f}{\partial y}\Big|_{(a,b)}\right|}_{\to 0}$

$P\to(a,b)$, $Q\to(a,b)$ as $(x,y)\to(a,b)$

$f: \mathbb{R}^m \to \mathbb{R}$, $f(x_1,\dots,x_m)$ is diff at $\vec{a} \in \mathbb{R}^m$

$\overset{\text{def}}{\iff}$ $f(\vec{x}) = f(\vec{a}) + \frac{\partial f}{\partial x_1}(\vec{a})\cdot(x_1-a_1) + \dots + \frac{\partial f}{\partial x_m}(\vec{a})\cdot(x_m-a_m) + o(\|\vec{x}-\vec{a}\|)$, $\vec{a} = (a_1,\dots,a_m)$.

$\vec{F}: \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $\vec{a}\in\mathbb{R}^m$, $\vec{a} = (a_1,\dots,a_m)$

$\vec{F} = (f_1(\underbrace{x_1,\dots,x_m}_{\vec{x}}), \dots, f_n(x_1,\dots,x_m))$

$\iff$ $\exists$ linear map $L: \mathbb{R}^m \to \mathbb{R}^n$, $L\vec{x} = \begin{bmatrix}\text{const} \\ \text{matrix}\end{bmatrix}\vec{x}$.

s.t. $\|\vec{F}(\vec{x}) - \vec{F}(\vec{a}) - L(\vec{x}-\vec{a})\| = o(\|\vec{x}-\vec{a}\|)$ as $\vec{x}\to\vec{a}$.

(i.e. $\vec{F}(\vec{x}) = \vec{F}(\vec{a}) + L(\vec{x}-\vec{a}) + o(\|\vec{x}-\vec{a}\|)$) $\quad \pi_j: \mathbb{R}^m \to \mathbb{R}$

$\iff |f_j(\vec{x}) - f_j(\vec{a}) - \pi_j\circ L(\vec{x}-\vec{a})| = o(\|\vec{x}-\vec{a}\|)$ $\forall j$.

Cor1 $\vec{F} = (f_1,\dots,f_n)$ is diff. at $\vec{a}$ $\iff$ f_j is diff. at $\vec{a}$ $\forall j$.

Cor2: $\pi_j\circ L(x_1,\dots,x_m)$ $\quad L: \mathbb{R}^m\to\mathbb{R}^n$, $\pi_j\circ L = \mathbb{R}^m \to \mathbb{R}^1$, $= A_1x_1 + \dots + A_mx_m$.

$= \frac{\partial f_j}{\partial x_1}\Big|_{\vec{a}}x_1 + \dots + \frac{\partial f_j}{\partial x_m}\Big|_{\vec{a}}x_m$

$\Rightarrow L\begin{bmatrix}x_1\\ \vdots \\ x_m\end{bmatrix} = \begin{bmatrix}\frac{\partial f_1}{\partial x_1}(\vec{a}) & \frac{\partial f_1}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial f_1}{\partial x_m}(\vec{a}) \\ \frac{\partial f_j}{\partial x_1}(\vec{a}) & \frac{\partial f_j}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial f_j}{\partial x_m}\Big|_{\vec{a}} \\ \frac{\partial f_n}{\partial x_1}(\vec{a}) & \cdots & \cdots & \frac{\partial f_n}{\partial x_m}(\vec{a})\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_m\end{bmatrix}$

$DF(\vec{a}) :=$ Jacobian matrix of $F = (f_j(x_1,\dots,x_m))$ $= \frac{\partial(f_1,\dots,f_n)}{\partial(x_1,\dots,x_m)}\Big|_{\vec{a}}$.

Chain rule:

$F: \mathbb{R}^m \to \mathbb{R}^n$, $G: \mathbb{R}^n \to \mathbb{R}^k$ differentiable

$G\circ F: \mathbb{R}^m \to \mathbb{R}^k$

$a \to F(\vec{a}) \to G(F(\vec{a}))$.

$D\vec{F}(\vec{a})$, $D\vec{G}(F(\vec{a}))$

$$\boxed{\begin{aligned} &D(G\circ F)(\vec{a}) \\ &= DG(F(\vec{a}))\underset{\text{matrix product}}{\cdot} DF(\vec{a})\end{aligned}}$$

$k\underset{DG}{\overset{n}{[\quad]}}\ \underset{DF(\vec{a})}{\overset{m}{[\quad]}}n$

$(y_1,y_2) = F(x_1,x_2)$, $(z_1,z_2) = G(y_1,y_2)$.

$\begin{bmatrix}\frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2}\end{bmatrix}$ $\quad DG = \begin{bmatrix}\frac{\partial z_1}{\partial y_1} & \frac{\partial z_1}{\partial y_2} \\ \frac{\partial z_2}{\partial y_1} & \frac{\partial z_2}{\partial y_2}\end{bmatrix}$

$DG\cdot DF = \begin{bmatrix}\frac{\partial z_1}{\partial y_1} & \frac{\partial z_1}{\partial y_2} \\ \frac{\partial z_2}{\partial y_1} & \frac{\partial z_2}{\partial y_2}\end{bmatrix}\begin{bmatrix}\frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2}\end{bmatrix}$

$= \begin{bmatrix}\frac{\partial z_1}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial z_1}{\partial y_2}\frac{\partial y_2}{\partial x_1} & \cdots \\ \cdots & \cdots\end{bmatrix}$

$= \begin{bmatrix}\frac{\partial z_1}{\partial x_1} & \cdots \\ \cdots & \cdots\end{bmatrix}$

$\vec{y} = \vec{F}(\vec{x}) = \vec{F}(\vec{a}) + D\vec{F}(\vec{a})\cdot(\vec{x}-\vec{a}) + o(\|\vec{x}-\vec{a}\|)$

$\vec{z} = \vec{G}(\vec{y}) = \vec{G}(\vec{F}(\vec{a})) + D\vec{G}(\vec{F}(\vec{a}))\cdot(\vec{y}-\vec{F}(\vec{a})) + o(\|\vec{y}-\vec{F}(\vec{a})\|)$

$\vec{h}(\vec{F}(\vec{x})) = \vec{G}(\vec{F}(\vec{a})) + \cdots$

If $\exists\vec{F}^{-1}$:

$\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ $\qquad \frac{\partial(x_1,\dots,x_n)}{\partial(x_1,\dots,x_n)}$

$\vec{F}^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ (diff.)

$\begin{cases}\vec{F}^{-1}\circ\vec{F} = id \\ \vec{F}\circ\vec{F}^{-1} = id\end{cases} \Rightarrow D(F^{-1}\circ F) = D(id) = id.$, $D(F^{-1}\circ F) = DF^{-1}\cdot DF$

$\Downarrow$

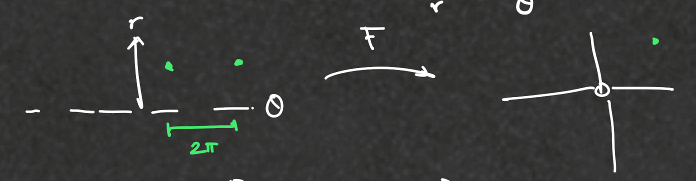
$DF\circ DF^{-1} = id.$

If $F: \mathbb{R}^n\to\mathbb{R}^n$ is diff. and $\exists F^{-1}: \mathbb{R}^n\to\mathbb{R}^n$ diff.

then: $(DF)^{-1} = DF^{-1}$ (in particular DF is invertible).

Converse? No!

$F(r,\theta) = (r\cos\theta, r\sin\theta)$: $\underset{r}{(0,\infty)}\times\underset{\theta}{\mathbb{R}} \to \mathbb{R}^2\setminus\{(0,0)\}$



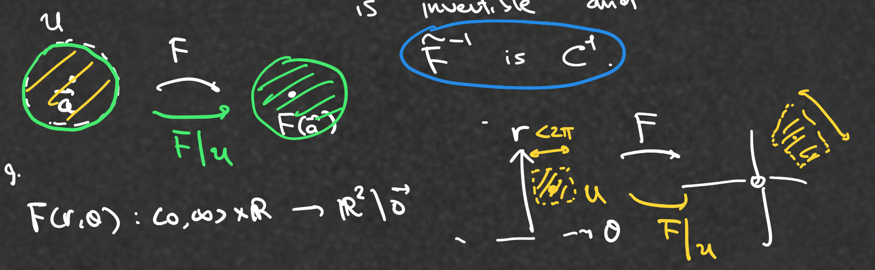
$DF(r,\theta) = \begin{bmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{bmatrix}$

$\det DF = r > 0.$

Inverse Function Theorem

Given: $\begin{cases}F: \mathbb{R}^n \to \mathbb{R}^n \quad C^1 \\ DF(\vec{a}) \text{ is invertible} \quad (\det DF(\vec{a}) \neq 0).\end{cases}$, $\vec{a}\in\mathbb{R}^n$

then $\exists\, U \ni \vec{a}$ open set $\subset\mathbb{R}^n$ s.t. $\tilde{F} := F|_U: U \to F(U)$ is invertible and $\tilde{F}^{-1}$ is C^1.



e.g. $F(r,\theta): (0,\infty)\times\mathbb{R} \to \mathbb{R}^2\setminus\vec{0}$

Banach's Contraction Mapping Theorem

$T: \mathbb{R}^n \to \mathbb{R}^n$ — can be replaced by any complete normed vector space. (Banach space)

Suppose $\exists\,\alpha\in(0,1)$ s.t.

$\|T(x) - T(y)\| \le \alpha\|x-y\|$ $\forall x,y\in\mathbb{R}^n$.

$\Rightarrow \exists!\ \bar{x}\in\mathbb{R}^n$ s.t. $T(\bar{x}) = \bar{x}$.

$x_n = T(x_{n-1})$

$F(a)$, y

if $\exists\,\bar{x} = \bar{x}(y)$, $\boxed{T_y(\bar{x}) = \bar{x}}$

$\Updownarrow$

$\cancel{\bar{x}} - A(y - F(\bar{x})) = \cancel{\bar{x}}$

$y = F(\bar{x})$.

$T_y(\vec{x}): \overline{B_\delta(a)} \to \overline{B_\delta(a)}$ (δ to be chosen)

$T_y(x) := x - A(y - F(x))$ (A: invertible matrix)

$\boxed{\|Ax\| \le \|A\|\,\|x\|}$ ← Exercise.

$[A] = [a_{ij}]$

$\|A\| = \left(\sum_{i,j=1}^{n} a_{ij}^2\right)^{1/2}$

$\|T_y(x_1) - T_y(x_2)\| = \|x_1 - A(\cancel{y} - F(x_1)) - x_2 + A(\cancel{y} - F(x_2))\|$

$= \|(x_1 - AF(x_1)) - (x_2 - AF(x_2))\|$

$= \|A(A^{-1}x_1 - F(x_1)) - A(A^{-1}x_2 - F(x_2))\|$

$\le \|A\|\,\|(A^{-1}x_1 - F(x_1)) - (A^{-1}x_2 - F(x_2))\|$

convex B: $\|G(x) - G(y)\| \le \max_{z\in B}\|DG(z)\|\cdot\|x-y\|$

$G(x) := A^{-1}Id - F$

$DG = A^{-1} - DF$

$\le \|A\|\cdot\max_{z\in\overline{B_\delta(a)}}\|A^{-1} - DF(z)\|\cdot\|x_1 - x_2\|$.

Choose $A = DF(\vec{a})^{-1}$

$A^{-1} - DF(z) = DF(a) - DF(z)$ as $z\to a$

$DF(z)$ is continuous

$\Rightarrow \exists\,\delta > 0$ s.t. $z\in\overline{B_\delta(a)} \Rightarrow \|DF(z) - DF(a)\| \le \frac{1}{777\|A\|}$

$\Rightarrow \|A\|\max_{B_\delta(a)}\|A^{-1} - DF(z)\| \le \frac{1}{777} < 1.$

$\Rightarrow \|T_y(x_1) - T_y(x_2)\| \le \frac{1}{777}\|x_1 - x_2\|$, $\forall x_1,x_2\in\overline{B_\delta(a)}$ (i.e. contraction!)

$\|T_y(x) - a\|$ $\qquad T_y: \overline{B_\delta(a)} \to \overline{B_\delta(a)}$ Want

$\le \|T_y(x) - T_y(a)\| + \|T_y(a) - a\|$

$\le \frac{1}{777}\underbrace{\|x-a\|}_{\le\delta} + \|\cancel{a} - DF(a)^{-1}(\vec{y} - F(\vec{a})) - \cancel{a}\| \le \delta.$ $\Bigg\}\Rightarrow T_y(\overline{B_\delta(a)}) \subset \overline{B_\delta(a)}$.

$\le \|DF(a)^{-1}\|\,\underbrace{\|y - F(a)\|}_{<\varepsilon} \le \frac{\delta}{2}$

$\varepsilon < \frac{\delta}{2}\frac{1}{\|DF^{-1}(a)\|}$.

$T_y: \overline{B_\delta(a)} \to \overline{B_\delta(a)}$ has a fixed point $\bar{x}(y)$.

$\Downarrow$

$y = F(\bar{x}(y))$

$\|\Phi_y(x_1) - \Phi_y(x_2)\| \le \frac{1}{777}\|x_1 - x_2\|$ $\quad\forall x_1,x_2\in B_\delta(a)$

Claim: F^{-1} is continuous.

$\|\underset{\sim}{x_1} - DF(a)^{-1}(y - F(x_1)) \underset{\sim}{- x_2} + DF(a)^{-1}(y - F(x_2))\| \le \frac{1}{2}\|x_1 - x_2\|$

$\|x_1 - x_2\| - \|DF(a)^{-1}(\cancel{y} - F(x_1)) - DF(a)^{-1}(\cancel{y} - F(x_2))\| \le$

$\le \frac{1}{2}\|x_1 - x_2\|$

$\|x-y\| \ge \|x\| - \|y\|$.

$\Rightarrow \frac{1}{2}\|x_1 - x_2\| \le \|DF(a)^{-1}(F(x_1) - F(x_2))\|$

$\le \|DF(a)^{-1}\|\,\|F(x_1) - F(x_2)\|$

$x_1 = F^{-1}(\underset{y}{F(x_1)})$ $\qquad \Downarrow$

$\frac{1}{2}\|F^{-1}(y_1) - F^{-1}(y_2)\| \le \|DF(a)^{-1}\|\,\|y_1 - y_2\|$ ∎

Claim F^{-1} is differentiable.

Proof: $\|F^{-1}(y) - F^{-1}(y_0) - (DF(\cancel{y_0}^{\,x_0}))^{-1}(y - y_0)\|$

$= \|F^{-1}(F(x)) - F^{-1}(F(x_0)) - DF(\cancel{y_0}^{\,x_0})^{-1}(F(x) - F(x_0))\|$

$= \|x - x_0 - DF(\cancel{y_0}^{\,x_0})^{-1}(F(x) - F(x_0))\|$

$= \|x - x_0 - DF(\cancel{y_0}^{\,x_0})^{-1}(DF(x_0)\cdot(x - x_0) + o(\|x - x_0\|))\|$

$= \|\cancel{(I - DF(y_0)^{-1}DF(x_0))}(x - x_0) + DF(\cancel{y_0}^{\,x_0})^{-1}o(\|x - x_0\|)\| \le o(\|y - y_0\|)$

 $o(\|x - x_0\|) = o(\|y - y_0\|)$ ∎ $\qquad\therefore F^{-1}$ is differentiable.

fct inv.

$(DF^{-1})(y_0) = (DF(x_0))^{-1}$ ← matrix inverse, $y_0 = F(x_0)$

$= \frac{1}{\det DF(x_0)}(\text{adj } DF(x_0))^{T?}$

$=$ rational function of $\frac{\partial f_i}{\partial x_j}$'s

(hence entries of DF^{-1} are continuous, i.e. F^{-1} is C^1).