Suppose $(U; u_1, \ldots, u_n)$ and $(V; v_1, \ldots, v_n)$ are two overlapping local coordinates of M,

and denote the local expressions of Ω and X by:

$$X = \sum_i X^i \frac{\partial}{\partial u_i} = \underbrace{\sum_\alpha \tilde{X}^\alpha \frac{\partial}{\partial v_\alpha}}_{= \sum_{i,\alpha} \left(\tilde{X}^\alpha \frac{\partial u_i}{\partial v_\alpha}\right) \frac{\partial}{\partial u_i}}$$

$$\Downarrow$$

$$\boxed{X^i = \sum_\alpha \tilde{X}^\alpha \frac{\partial u_i}{\partial v_\alpha}} \Longleftarrow \sum_{i,\alpha} \left(\tilde{X}^\alpha \frac{\partial u_i}{\partial v_\alpha}\right) \frac{\partial}{\partial u_i}$$

$$\Omega = e^{f_u} du^1 \wedge \cdots \wedge du^n = \underbrace{e^{f_v} dv^1 \wedge \cdots \wedge dv^n}_{= e^{f_v} \det \frac{\partial(v_1, \ldots, v_n)}{\partial(u_1, \ldots, u_n)} du^1 \wedge \cdots \wedge du^n}$$

$$\Downarrow$$

$$e^{f_u} = e^{f_v} \det \frac{\partial(v_1, \ldots, v_n)}{\partial(u_1, \ldots, u_n)} \Longleftarrow e^{f_v} \det \frac{\partial(v_1, \ldots, v_n)}{\partial(u_1, \ldots, u_n)} du^1 \wedge \cdots \wedge du^n$$

$$\Downarrow$$

$$\boxed{f_u = f_v + \log \det \frac{\partial(v_1, \ldots, v_n)}{\partial(u_1, \ldots, u_n)}}$$

We need to show:

$$\sum_i \left(\frac{\partial X^i}{\partial u_i} + X^i \frac{\partial f_u}{\partial u_i} \right) = \sum_\alpha \left(\frac{\partial \tilde{X}^\alpha}{\partial v_\alpha} + \tilde{X}^\alpha \frac{\partial f_v}{\partial v_\alpha} \right)$$

$$\sum_i \left(\frac{\partial X^i}{\partial u_i} + X^i \frac{\partial f_u}{\partial u_i} \right)$$

$$= \sum_i \left[\sum_\alpha \frac{\partial v_\alpha}{\partial u_i}\frac{\partial}{\partial v_\alpha}\left(\sum_\beta \tilde X^\beta \frac{\partial u_i}{\partial v_\beta}\right) + \sum_\beta \tilde X^\beta \frac{\partial u_i}{\partial v_\beta}\cdot \sum_\alpha \frac{\partial v_\alpha}{\partial u_i}\frac{\partial}{\partial v_\alpha}\left(f_v + \log\det \underbrace{\frac{\partial(v)}{\partial(u)}}_{\frac{\partial(v_1,\dots,v_n)}{\partial(u_1,\dots,u_n)}}\right)\right]$$

$$= \sum_{i,\alpha,\beta} \frac{\partial v_\alpha}{\partial u_i}\frac{\partial \tilde X^\beta}{\partial v_\alpha}\cdot\frac{\partial u_i}{\partial v_\beta} + \sum_{i,\alpha,\beta}\frac{\partial v_\alpha}{\partial u_i}\tilde X^\beta \frac{\partial^2 u_i}{\partial v_\alpha \partial v_\beta}$$

$$+ \sum_{i,\alpha,\beta}\tilde X^\beta \frac{\partial u_i}{\partial v_\beta}\frac{\partial v_\alpha}{\partial u_i}\frac{\partial f_v}{\partial v_\alpha} + \sum_{i,\alpha,\beta}\tilde X^\beta\frac{\partial u_i}{\partial v_\beta}\frac{\partial v_\alpha}{\partial u_i}\frac{\partial}{\partial v_\alpha}\log\det\frac{\partial(v)}{\partial(u)}$$

chain rule

$$= \sum_\alpha \frac{\partial \tilde X^\alpha}{\partial v_\alpha} + \sum_{i,\alpha,\beta}\frac{\partial v_\alpha}{\partial u_i}\tilde X^\beta\frac{\partial^2 u_i}{\partial v_\alpha\partial v_\beta} \quad \text{— } (*)$$

$$+ \sum_\alpha \tilde X^\alpha \frac{\partial f_v}{\partial v_\alpha} + \sum_\alpha \tilde X^\alpha \frac{\partial}{\partial v_\alpha}\log\det\frac{\partial(v)}{\partial(u)} \quad \text{— } (**)$$

$$\frac{\partial}{\partial v_\alpha}\log\det\frac{\partial(v)}{\partial(u)} = \frac{\partial}{\partial v_\alpha}\log\frac{1}{\det\frac{\partial(u)}{\partial(v)}} \qquad \text{as } \left(\frac{\partial(u)}{\partial(v)}\right)^{-1} = \frac{\partial(v)}{\partial(u)}$$

$$= -\frac{\partial}{\partial v_\alpha}\log\det\frac{\partial(u)}{\partial(v)} = -\mathrm{Tr}\left(\frac{\partial(u)}{\partial(v)}^{-1}\cdot\frac{\partial}{\partial v_\alpha}\left(\frac{\partial(u)}{\partial(v)}\right)\right)$$

$$= -\sum_{i,j}\left(\frac{\partial(u)}{\partial(v)}\right)^{-1}_{ij}\cdot\frac{\partial}{\partial v_\alpha}\left(\frac{\partial u_j}{\partial v_i}\right) = -\sum_{i,j}\frac{\partial v_i}{\partial u_j}\frac{\partial^2 u_j}{\partial v_i\partial v_\alpha}$$

$$= -\sum_{j,\beta}\frac{\partial^2 u_j}{\partial v_\alpha\partial v_\beta}\frac{\partial v_\beta}{\partial u_j}$$

$\therefore\ (*) + (**) = 0$

∎