MATH 4033 • Spring 2017 • Calculus on Manifolds
Problem Set #5 • de Rham Cohomology • Due Date: 08/05/2017, 10:20AM

Instruction: Turn in Problems #1 - #3. You are recommended to do Problems #4 - #5 before the final exam. If you wish, try Problem #6 after the final exam.

1. The purpose of this exercise is to prove that $H^2(\mathbb{R}^3) = 0$, i.e. every closed 2-form on $\mathbb{R}^3$ must be exact. Consider a closed form:

$$\omega = A\,dy \wedge dz + B\,dz \wedge dx + C\,dx \wedge dy$$

where A, B and C are smooth scalar functions of (x, y, z). Define the following 1-form:

$$\begin{aligned}\alpha := &\left(\int_0^1 A(tx, ty, tz)t\,dt\right)(y\,dz - z\,dy)\\ &+ \left(\int_0^1 B(tx, ty, tz)t\,dt\right)(z\,dx - x\,dz)\\ &+ \left(\int_0^1 C(tx, ty, tz)t\,dt\right)(x\,dy - y\,dx)\end{aligned}$$

First, compute $d\alpha$; then use the result to show that ω is exact.

Solution: ω is closed implies

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = 0.$$

Next, we have

$$\begin{aligned}\alpha = &\left(\int_0^1 A(tx, ty, tz)t\,dt\right)(y\,dz - z\,dy)\\ &+ \left(\int_0^1 B(tx, ty, tz)t\,dt\right)(z\,dx - x\,dz)\\ &+ \left(\int_0^1 C(tx, ty, tz)t\,dt\right)(x\,dy - y\,dx)\\ = &\left(\int_0^1 B(tx, ty, tz)tz - C(tx, ty, tz)ty\,dt\right)dx\\ &+ \left(\int_0^1 C(tx, ty, tz)tx - A(tx, ty, tz)tz\,dt\right)dy\\ &+ \left(\int_0^1 A(tx, ty, tz)ty - B(tx, ty, tz)tx\,dt\right)dz\end{aligned}$$

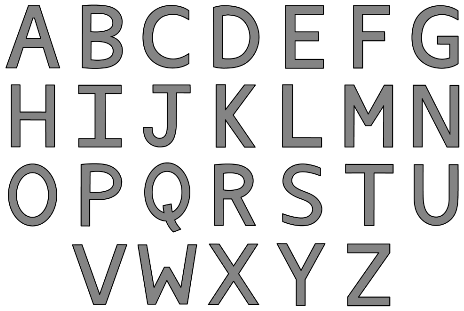
$$
\begin{aligned}
d\alpha = &\left(\int_0^1 t \frac{\partial}{\partial y} \big(B(tx,ty,tz)z - C(tx,ty,tz)y \big)\, dt \right) dy \wedge dx \\
&+ \left(\int_0^1 t \frac{\partial}{\partial z} \big(B(tx,ty,tz)z - C(tx,ty,tz)y \big)\, dt \right) dz \wedge dx \\
&+ \left(\int_0^1 t \frac{\partial}{\partial x} \big(C(tx,ty,tz)x - A(tx,ty,tz)z \big)\, dt \right) dx \wedge dy \\
&+ \left(\int_0^1 t \frac{\partial}{\partial z} \big(C(tx,ty,tz)x - A(tx,ty,tz)z \big)\, dt \right) dz \wedge dy \\
&+ \left(\int_0^1 t \frac{\partial}{\partial y} \big(A(tx,ty,tz)y - B(tx,ty,tz)x \big)\, dt \right) dy \wedge dz \\
&+ \left(\int_0^1 t \frac{\partial}{\partial x} \big(A(tx,ty,tz)y - B(tx,ty,tz)x \big)\, dt \right) dx \wedge dz.
\end{aligned}
$$

From now on, just for simplicity, we abbreviate $A(tx,ty,tz)$ by just A. Similarly for B and C. Continuation on our computation using the chain rule, we get:

$$
\begin{aligned}
&= \left(\int_0^1 t^2 \left(x\frac{\partial C}{\partial x} + y\frac{\partial C}{\partial y} + z\frac{\partial C}{\partial z} \right) - t^2 z \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \right) + 2tC(tx,ty,tz)\, dt \right) dx \wedge dy \\
&+ \left(\int_0^1 t^2 \left(x\frac{\partial A}{\partial x} + y\frac{\partial A}{\partial y} + z\frac{\partial A}{\partial z} \right) - t^2 x \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \right) + 2tA(tx,ty,tz)\, dt \right) dy \wedge dz \\
&+ \left(\int_0^1 t^2 \left(x\frac{\partial B}{\partial x} + y\frac{\partial B}{\partial y} + y\frac{\partial B}{\partial z} \right) - t^2 y \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \right) + 2tB(tx,ty,tz)\, dt \right) dz \wedge dx \\
&= \left(\int_0^1 t^2 \left(x\frac{\partial C}{\partial x} + y\frac{\partial C}{\partial y} + z\frac{\partial C}{\partial z} \right) + 2tC(tx,ty,tz)\, dt \right) dx \wedge dy \\
&\quad + \left(\int_0^1 t^2 \left(x\frac{\partial A}{\partial x} + y\frac{\partial A}{\partial y} + x\frac{\partial A}{\partial z} \right) + 2tA(tx,ty,tz)\, dt \right) dy \wedge dz \\
&\quad + \left(\int_0^1 t^2 \left(x\frac{\partial B}{\partial x} + y\frac{\partial B}{\partial y} + y\frac{\partial B}{\partial z} \right) + 2tB(tx,ty,tz)\, dt \right) dz \wedge dx \\
&= \left(\int_0^1 \frac{d}{dt} \big(C(tx,ty,tz)t^2 \big)\, dt \right) dx \wedge dy \\
&\quad + \left(\int_0^1 \frac{d}{dt} \big(A(tx,ty,tz)t^2 \big)\, dt \right) dy \wedge dz \\
&\quad + \left(\int_0^1 \frac{d}{dt} \big(B(tx,ty,tz)t^2 \big)\, dt \right) dz \wedge dx \\
&= A\, dy \wedge dz + B\, dz \wedge dx + C\, dx \wedge dy \\
&= \omega.
\end{aligned}
$$

Thus, ω is exact.

2. Consider the following alphabet. Each letter is regarded as a solid region.



Answer the following without justification:

(a) Which letter(s) is/are contractible?

Solution: C, E, F, G, H, I, J, K, L, M,N, S, T, U, V, W, X, Y, Z.

(b) Which letter(s) is/are star-shaped?

Solution: K, L, T, V, X, Y.

(c) Which letter(s) has/have non-zero 1st Betti number b_1?

Solution: A, B, D, O, P, Q, R.

3. Prove the following statements about deformation retracts by explicitly constructing Ψ_t.

(a) Show that the Möbius strip Σ defined in Example 4.11 deformation retracts onto a circle. [Hence, $H^1_{\text{dR}}(\Sigma) = H^1_{\text{dR}}(\mathbb{S}^1) = \mathbb{R}$.]

Solution: Let

$$\mathsf{F} : (-1,1) \times (0,2\pi) \to \Sigma \qquad \widetilde{\mathsf{F}} : (-1,1) \times (-\pi,\pi) \to \Sigma$$

$$\mathsf{F}(u,\theta) = \begin{bmatrix} \left(3 + u\cos\frac{\theta}{2}\right)\cos\theta \\ \left(3 + u\cos\frac{\theta}{2}\right)\sin\theta \\ u\sin\frac{\theta}{2} \end{bmatrix} \qquad \widetilde{\mathsf{F}}(\widetilde{u},\widetilde{\theta}) = \begin{bmatrix} \left(3 + \widetilde{u}\cos\frac{\widetilde{\theta}}{2}\right)\cos\widetilde{\theta} \\ \left(3 + \widetilde{u}\cos\frac{\widetilde{\theta}}{2}\right)\sin\widetilde{\theta} \\ \widetilde{u}\sin\frac{\widetilde{\theta}}{2} \end{bmatrix}$$

be two local parametrizations of Σ which cover Σ.
Define $\{\Psi_t : \Sigma \to \Sigma\}_{t\in[0,1]}$ be a C^1 family of smooth maps as

$$\begin{cases} \Psi_t\big(\mathsf{F}(u,\theta)\big) = \mathsf{F}\big((1-t)u,\theta\big) \\ \Psi_t\big(\widetilde{\mathsf{F}}(\widetilde{u},\widetilde{\theta})\big) = \widetilde{\mathsf{F}}\big((1-t)\widetilde{u},\widetilde{\theta}\big) \end{cases}.$$

Then, it is easy to see that

$$\begin{cases} \Psi_0\big(\mathsf{F}(u,\theta)\big) = \mathsf{F}(u,\theta) \\ \Psi_0\big(\widetilde{\mathsf{F}}(\widetilde{u},\widetilde{\theta})\big) = \widetilde{\mathsf{F}}(\widetilde{u},\widetilde{\theta}) \end{cases},$$

$$\begin{cases} \Psi_1\big(\mathsf{F}(u,\theta)\big) = \mathsf{F}(0,\theta) = \begin{bmatrix} 3\cos\theta & 3\sin\theta & 0 \end{bmatrix}^T \\ \Psi_1\big(\widetilde{\mathsf{F}}(\widetilde{u},\widetilde{\theta})\big) = \widetilde{\mathsf{F}}(0,\widetilde{\theta}) = \begin{bmatrix} 3\cos\widetilde{\theta} & 3\sin\widetilde{\theta} & 0 \end{bmatrix}^T \end{cases},$$

$$\begin{cases} \Psi_t\big(\mathsf{F}(0,\theta)\big) = \mathsf{F}(0,\theta) \\ \Psi_t\big(\widetilde{\mathsf{F}}(0,\widetilde{\theta})\big) = \widetilde{\mathsf{F}}(0,\widetilde{\theta}) \end{cases}.$$

Hence, the Möbius strip Σ defined in Example 4.11 deformation retracts onto a circle with radius 3. Since a circle with radius 3 is diffeomorphic to $\mathbb{S}^1$, we have $H^1_{\text{dR}}(\Sigma) = H^1_{\text{dR}}(\mathbb{S}^1) = \mathbb{R}$.

(b) The zero section Σ_0 of the tangent bundle TM of a smooth manifold M is defined to be:

$$\Sigma_0 := \{(p, 0_p) \in p \times T_pM : p \in M\}$$

where 0_p is the zero vector in T_pM. Show that Σ_0 is a deformation retract of TM. [Hence, $H^*_{\text{dR}}(TM) = H^*_{\text{dR}}(\Sigma_0) = H^*_{\text{dR}}(M)$.]

Solution: Let $\{F_i(u_1, \cdots, u_n) : U_i \to O_i\}$ be a family of local parametrizations of M which covers M, the induced local parametrization $\widetilde{F}_i : U_i \times \mathbb{R}^n \to TM$ of the tangent bundle TM is

$$\widetilde{F}_i(u_1, \cdots, u_n, a^1, \cdots, a^n) = \left(F_i(u_1, \cdots, u_n), a^1\frac{\partial}{\partial u_1} + \cdots + a^n\frac{\partial}{\partial u_n}\right) \in TM$$

and

$$\Sigma_0 = \bigcup_i \widetilde{F}_i(u_1, \cdots, u_n, 0, \cdots, 0).$$

Define $\{\Psi_t : TM \to TM\}_{t\in[0,1]}$ be a C^1 family of smooth maps as

$$\Psi_t\Big(\widetilde{F}_i(u_1, \cdots, u_n, a^1, \cdots, a^n)\Big) = \widetilde{F}_i\big(u_1, \cdots, u_n, (1-t)a^1, \cdots, (1-t)a^n\big).$$

Then for any i, we have

$$\begin{aligned} \Psi_0\Big(\widetilde{F}_i(u_1, \cdots, u_n, a^1, \cdots, a^n)\Big) &= \widetilde{F}_i\big(u_1, \cdots, u_n, a^1, \cdots, a^n\big), \\ \Psi_1\Big(\widetilde{F}_i(u_1, \cdots, u_n, a^1, \cdots, a^n)\Big) &= \widetilde{F}_i\big(u_1, \cdots, u_n, 0, \cdots, 0\big), \\ \Psi_t\Big(\widetilde{F}_i(u_1, \cdots, u_n, 0, \cdots, 0)\Big) &= \widetilde{F}_i\big(u_1, \cdots, u_n, 0, \cdots, 0\big). \end{aligned}$$

Thus, Σ_0 is a deformation retract of TM.

In the following problems, you may assume the Poincaré's Lemma and Deformation Retract Invariance hold on any H^k. Also, we may use the following fact without proof:

On a compact, connected orientable manifold M without boundary, then:

- $\dim H^n(M) = 1$ where $n = \dim M$
- $H^n(M\backslash\{p\}) = 0$ for any $p \in M$.

4. Let $\mathbb{T}^2$ be the 2-dimensional torus. Show that $b_1(\mathbb{T}^2) = 2$.

Solution: Divide the torus horizontally and "fatten" each half-torus a bit. Take U to be the upper-half, and V to be the lower-half.

Then each of U and V are diffeomorphic to an annulus, which can be deformation retracted onto a circle. Hence:

$$\dim H^0(U) = 1, \quad \dim H^1(U) = 1, \quad \dim H^2(U) = 0.$$

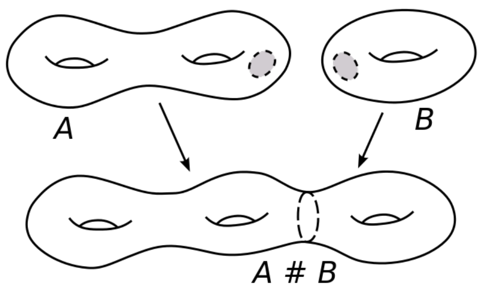
The same for V.

Observe that $U \cap V$ is a disjoint union of two thin cylinders, so

$$\dim H^0(U \cap V) = 2, \quad \dim H^1(U \cap V) = 2, \quad \dim H^2(U \cap V) = 0.$$

Putting these into the Mayer-Vietoris sequence can consider the alternating sum, and combining with the given fact that $\dim H^2(\mathbb{T}^2) = 1$, one can then conclude $\dim H^1(\mathbb{T}^2) = 2$ easily.

5. Given two compact smooth 2-manifolds A and B without boundary, its connected sum $A\#B$ is a 2-manifold obtained by removing an open ball in each of A and B, and then gluing them along the two boundary circles:



(a) Show that $A\#B$ is orientable if both A and B are so. [Hint: use partitions of unity to construct a global non-vanishing 2-form.]

Solution: Given that A is orientable, it has an oriented atlas (whose transition maps have positive Jacobian determinant). Its subset $A\backslash B_\varepsilon(p)$ is also orientable since one can parametrize it by the atlas induced from that of A. The induced atlas is clearly orientable since the transition maps are simply restrictions of those in the atlas of A. Similarly, $B\backslash B_\varepsilon(q)$ is also orientable.

Now regard $A\backslash B_\varepsilon(p)$ and $B\backslash B_\varepsilon(q)$ are subsets of $A\#B$ by "fattening" them if necessary. Let Ω_A and Ω_B be the orientation 2-forms of $A\backslash B_\varepsilon(p)$ and $B\backslash B_\varepsilon(q)$ respectively.

WLOG, we assume that $\Omega_A(\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_n})$ and $\Omega_B(\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_n})$, where $(u_1, \ldots, u_n)$ are local coordinates on the overlap between $A \backslash B_\varepsilon(p)$ and $B \backslash B_\varepsilon(q)$, have the same sign on the overlap.
Let $\{\rho_A, \rho_B\}$ be a partitions of unity subordinate to $\{A \backslash B_\varepsilon(p), B \backslash B_\varepsilon(q)\}$, then construct:

$$\Omega := \rho_A \Omega_A + \rho_B \Omega_B.$$

It is then a smooth non-vanishing 2-form on $A \# B$. It is non-vanishing because at each point $x \in A \# B$, at least one of $\rho_A(x)$ and $\rho_B(x)$ is non-zero.

(b) Using Mayer-Vietoris sequence, show that $b_1(A \# B) = b_1(A) + b_1(B)$.

Solution: (Sketch) First apply Mayer-Vietoris to show that $b_1(A) = b_1(A \backslash B_\varepsilon(p))$ and similarly for $b_1(B)$. To prove this, pick $U = A \backslash \{p\}$ and $V = B_\varepsilon(p)$, then $U \cap V$ is an annulus. From the given fact that $H^2(U) = 0$, one can show $b_1(A) = b_1(A \backslash B_\varepsilon(p))$ by considering an alternating sum in the Mayer-Vietoris sequence.
Secondly, apply Mayer-Vietoris again with $U = A \backslash B_\varepsilon(p) \subset A \# B$ and $V = B \backslash B_\varepsilon(q) \subset A \# B$, then $U \cap V$ is an annulus. The rest follows by considering an alternating sum of the Mayer-Vietoris sequence.

6. (∞ points (bonus)) Prove or disprove: "Every Hodge cohomology class of a non-singular complex projective manifold $X \subset \mathbb{CP}^N$ is a linear combination with rational coefficients of the cohomology classes of complex subvarieties of X."

End of all MATH 4033 homework.
"The chain will be broken and all men will have their reward."
(from *Les Misérables*)