# Problem Set #3

MATH 4033, Calculus on Manifold, Spring 2019

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**Problem 1.** (a) Show that any complex manifold (i.e. transition maps are holomorphic) must be orientable.

**Solution.** Let $F(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and $G(u_1, \ldots, u_n, v_1, \ldots, v_n)$ be local parametrizations on a complex manifold $M$. Since the transition map

$$(u_1, \ldots, u_n, v_1, \ldots, v_n) = G^{-1} \circ F(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

is holomorphic, the Cauchy-Riemann equations must be satisfied: $\forall 1 \leq i, j \leq n$,

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial v_i}{\partial y_j} \qquad \text{and} \qquad \frac{\partial v_i}{\partial x_j} = -\frac{\partial u_i}{\partial y_j}$$

Thus the Jacobian of the transition map is given by the block matrix:

$$D(G^{-1} \circ F) = \left[\begin{array}{c|c} \dfrac{\partial u_i}{\partial x_j} & \dfrac{\partial u_i}{\partial y_j} \\ \hline \dfrac{\partial v_i}{\partial x_j} & \dfrac{\partial v_i}{\partial y_j} \end{array}\right] = \left[\begin{array}{c|c} \dfrac{\partial u_i}{\partial x_j} & \dfrac{\partial u_i}{\partial y_j} \\ \hline -\dfrac{\partial u_i}{\partial y_j} & \dfrac{\partial u_i}{\partial x_j} \end{array}\right]$$

To compute its determinant we will need a formula from linear algebra to compute the determinant of block matrices of the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

This is achieved by using the relation

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \frac{1}{2} \begin{pmatrix} I & I \\ iI & -iI \end{pmatrix} \begin{pmatrix} A + iB & \\ & A - iB \end{pmatrix} \begin{pmatrix} I & -iI \\ I & iI \end{pmatrix}$$

Here $I = I_n$ is the $n \times n$ identity matrix, while $i = \sqrt{-1}$. Note that

$$\begin{pmatrix} I & I \\ iI & -iI \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} I & -iI \\ I & iI \end{pmatrix}$$

Therefore,

$$\begin{aligned} \det \begin{pmatrix} A & B \\ -B & A \end{pmatrix} &= \det \begin{pmatrix} A + iB & \\ & A - iB \end{pmatrix} \\ &= \det(A + iB) \det(A - iB) \\ &= \det(A + iB) \det(\overline{A + iB}) \\ &= \det(A + iB) \overline{\det(A + iB)} \geq 0 \end{aligned}$$

Since $\det D(G^{-1} \circ F) \neq 0$, thus it must follow that

$$\det D(G^{-1} \circ F) = \det \left[ \begin{array}{c|c} \dfrac{\partial u_i}{\partial x_j} & \dfrac{\partial u_i}{\partial y_j} \\ \hline -\dfrac{\partial u_i}{\partial y_j} & \dfrac{\partial u_i}{\partial x_j} \end{array} \right] > 0$$

Hence by definition, $M$ is orientable. $\square$

(b) A smooth manifold $M^{2n}$ is called a symplectic manifold if there exists a smooth 2-form $\omega$ such that $d\omega = 0$ and the only vector field $X$ such that $\iota_X \omega = 0$ is the zero vector field. Show that any symplectic manifold must be orientable.

**Solution.** Consider the $2n$-form

$$\Omega := \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}}$$

We will show that $\Omega$ is a non-vanishing form on $M$. By **Proposition 4.25**, this will imply that $M$ is orientable.

Let $p \in M$. Let $(u_1, \ldots, u_n)$ be local coordinates around $p$. Then we can write

$$\omega_p = \sum_{i,j=1}^{n} \omega_{ij}(p) du^i \wedge du^j$$

Since $\omega$ is a differential form, it is alternating, thus the matrix $[\omega_{ij}(p)]$ is a real skew-symmetric matrix. Moreover, the assumption "if $\iota_X \omega = 0$, then $X = 0$" implies that the linear map $X_p \mapsto (\iota_X \omega)_p$ is injective (hence bijective since $T_pM$ and $T_p^*M$ have the same finite dimension). Thus, $[\omega_{ij}(p)]$ is an invertible matrix. Therefore, by results in linear algebra, $[\omega_{ij}(p)]$ can be decomposed into

$$[\omega_{ij}(p)] = Q \begin{pmatrix} \lambda_1 J & & \\ & \ddots & \\ & & \lambda_n J \end{pmatrix} Q^T$$

where $Q$ is orthogonal, $J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, and $\lambda_i > 0$ such that $\pm \lambda_i \sqrt{-1}$ are the eigenvalues of $[\omega_{ij}(p)]$. Consequently, there exists a basis $\{e_i^*\}_{i=1}^{2n}$ for $T_p^*M$ such that

$$\omega_p = \sum_{k=1}^{n} \lambda_k e_{2k-1}^* \wedge e_{2k}^*$$

which implies that

$$\Omega_p = \omega_p \wedge \cdots \wedge \omega_p = n! \, (\lambda_1 \cdots \lambda_n) \, e_1^* \wedge \cdots \wedge e_{2n}^* \neq 0$$

since $\lambda_1 \cdots \lambda_n > 0$. Since $\Omega_p \neq 0$ is true for any $p \in M$, we conclude that $\Omega$ is a non-vanishing $2n$-form on $M$. This completes the proof. $\square$

**Problem 2.** Let $M^n$ be a smooth manifold. For each $p \in M$ covered by local coordinates $(u_1, \ldots, u_n)$, we denote

$$\wedge^n T_p^* M := \text{span}\left\{ du^1\big|_p \wedge \cdots \wedge du^n\big|_p \right\}.$$

Denote the $n$-form bundle of $M$ by $\wedge^n T^* M := \bigcup_{p \in M} \{p\} \times \wedge^n T_p^* M$.

(a) Show that the $n$-form bundle of $M$ is a smooth manifold. What is its dimension?

**Solution.** Denote the local parametrization around $p$ by $F(u_1, \ldots, u_n) : \mathcal{U}_F \to \mathcal{O}_F$. Define

$$\begin{aligned} \tilde{F} : \mathcal{U}_F \times \mathbb{R} &\to \pi^{-1}(\mathcal{O}_F) \\ (u_1, \ldots, u_n, a) &\mapsto \Big( F(u_1, \ldots, u_n), a(du^1 \wedge \cdots \wedge du^n)\big|_{F(u_1, \ldots, u_n)} \Big) \end{aligned}$$

where $\pi : \wedge^n T^* M \to M$ is the projection $(p, \omega_p) \mapsto p$, and for convenience, we denote $(du^1 \wedge \cdots \wedge du^n)\big|_p = du^1\big|_p \wedge \cdots \wedge du^n\big|_p$. Then $\tilde{F}$ is a local parametrization on $\wedge^n T^* M$.

Let $G(v_1, \ldots, v_n) : \mathcal{U}_G \to \mathcal{O}_G$ be another local parametrization around $p$, and let $\tilde{G}$ be the similarly induced local parametrization on $\wedge^n T^* M$. We consider the transition maps:

$$(v_1, \ldots, v_n, b) = \tilde{G}^{-1} \circ \tilde{F}(u_1, \ldots, u_n, a)$$

Then we have $\tilde{G}(v_1, \ldots, v_n, b) = \tilde{F}(u_1, \ldots, u_n, a)$, and so by definition of $\tilde{F}, \tilde{G}$, we have

$$\begin{aligned} G(v_1, \ldots, v_n) &= F(u_1, \ldots, u_n) \\ b(dv^1 \wedge \cdots \wedge dv^n)\big|_{G(v_1, \ldots, v_n)} &= a(du^1 \wedge \cdots \wedge du^n)\big|_{F(u_1, \ldots, u_n)} \end{aligned}$$

Fron these we also see that

$$\begin{aligned} b(dv^1 \wedge \cdots \wedge dv^n)\big|_{G(v_1, \ldots, v_n)} &= a \frac{\partial(u_1, \ldots, u_n)}{\partial(v_1, \ldots, v_n)} (dv^1 \wedge \cdots \wedge dv^n)\big|_{G(v_1, \ldots, v_n)} \\ \therefore \qquad b &= a \det D(F^{-1} \circ G) \\ &= \frac{a}{\det D(G^{-1} \circ F)} \end{aligned}$$

since $\det D(G^{-1} \circ F) \neq 0$ on $\mathcal{V} := \mathcal{U}_F \cap \mathcal{U}_G$. Therefore,

$$\tilde{G}^{-1} \circ \tilde{F}(u_1, \ldots, u_n, a) = \left( G^{-1} \circ F(u_1, \ldots, u_n), \frac{a}{\det D(G^{-1} \circ F)} \right)$$

Since $M$ is a smooth manifold, thus $G^{-1} \circ F$ is smooth. The last component is a rational function on $\mathcal{V}$, thus it is also smooth. Hence, $\tilde{G}^{-1} \circ \tilde{F}$.

Hence by definition, $\wedge^n T^* M$ is a smooth manifold of dimension $\dim \wedge^n T^* M = n + 1$. $\square$

(b) Show that if $M^n$ is orientable, then $\wedge^n T^*M$ is diffeomorphic to $M \times \mathbb{R}$.

**Solution.** By **Proposition 4.25** in lecture notes, it follows that there exists a non-vanishing smooth $n$-form $\Omega$ globally defined on $M$. Then for each $p \in M$, $\Omega_p$ is nonzero. Since $\wedge^n T_p^*M$ is 1-dimensional, thus

$$\wedge^n T_p^*M = \text{span}\,\{\Omega_p\} \tag{1}$$

Define $\Phi : M \times \mathbb{R} \to \wedge^n T^*M$ by

$$\Phi(p, \lambda) := (p, \lambda\Omega_p)$$

Since $\Omega$ is non-vanishing, this map $\Phi$ is injective. Indeed, if $(p, \lambda\Omega_p) = (q, \mu\Omega_q)$, then $p = q$ and $\lambda\Omega_p = \nu\Omega_p$ implies that $\lambda = \mu$ (since $\Omega_p \neq 0$). Finally, as an immediate consequence of (1), $\Phi$ is also surjective. Hence, $\Phi$ is a bijection.

Let $p \in M$. Let $F(u_1, \ldots, u_n) : \mathcal{U} \to \mathcal{O}$ be a local parametrization around $p$, and let $\tilde{F}$ be the induced local parametrization on $\wedge^n T^*M$ defined in part (a). Consider

$$\begin{aligned}(v_1, \ldots, v_n, a) &= \tilde{F}^{-1} \circ \Phi \circ (F \times id)(u_1, \ldots, u_n, \lambda)\\ &= \tilde{F}^{-1} \circ \Phi\,(F(u_1, \ldots, u_n), \lambda)\\ &= \tilde{F}^{-1}\left(F(u_1, \ldots, u_n), \lambda\Omega\big|_{F(u_1,\ldots,u_n)}\right)\end{aligned}$$

On $\mathcal{O}$, we may write

$$\Omega = f du^1 \wedge \cdots \wedge du^n$$

for some locally defined $f \in C^\infty(\mathcal{O})$. Then it follows that

$$a = \lambda f(F(u_1, \ldots, u_n))$$

and so

$$\tilde{F}^{-1} \circ \Phi \circ (F \times id)(u_1, \ldots, u_n, \lambda) = \big(u_1, \ldots, u_n, \lambda(f \circ F)(u_1, \ldots, u_n)\big)$$

Since $f$ is smooth on $\mathcal{O}$, thus $f \circ F$ is smooth on $\mathcal{U}$, therefore $\tilde{F}^{-1} \circ \Phi \circ (F \times id)$. is smooth. Hence, $\Phi$ is smooth.

Finally, since $\Omega$ is non-vanishing, thus $f$ is also non-vanishing on $\mathcal{U}$. Consequently we have

$$(F \times id)^{-1} \circ \Phi^{-1} \circ \tilde{F}(v_1, \ldots, v_n, a) = \left(v_1, \ldots, v_n, \frac{a}{(f \circ F)(v_1, \ldots, v_n)}\right)$$

and so $\Phi^{-1}$ is also smooth.

Hence by definition, $\Phi$ is a diffeomorphism. $\square$

**Problem 3.** Let $\omega$ be the $n$-form on $\mathbb{R}^{n+1} \setminus \{\vec{0}\}$ defined by:

$$\omega = \frac{1}{|\vec{x}|^{n+1}} \sum_{i=1}^{n+1} (-1)^i x_i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{n+1}$$

where $\vec{x} = (x_1, \ldots, x_{n+1})$ and $|\vec{x}| = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$.

(a) Show that $\omega$ is closed.

**Solution.** By direct computation, we have

$$\begin{aligned}
d\omega &= \sum_{i=1}^{n} (-1)^{i-1} \left( \sum_{j=1}^{n+1} \frac{\partial}{\partial x^j} \frac{x_i}{|\vec{x}|^{n+1}} \right) dx^j \wedge dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{n+1} \\
&= \sum_{i=1}^{n} (-1)^{i-1} \left( \frac{\partial}{\partial x^i} \frac{x_i}{|\vec{x}|^{n+1}} \right) dx^i \wedge dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{n+1} \\
&= \sum_{i=1}^{n} \left( \frac{1}{|\vec{x}|^{n+1}} - \frac{n+1}{2} \cdot \frac{2x_i^2}{|\vec{x}|^{n+3}} \right) dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^i \wedge dx^{i+1} \wedge \cdots \wedge dx^{n+1} \\
&= \left[ \sum_{i=1}^{n} \frac{1}{|\vec{x}|^{n+1}} - \frac{n+1}{|\vec{x}|^{n+3}} \sum_{i=1}^{n} x_i^2 \right] dx^1 \wedge \cdots \wedge dx^n \\
&= \left( \frac{n+1}{|\vec{x}|^{n+1}} - \frac{(n+1)|\vec{x}|^2}{|\vec{x}|^{n+3}} \right) dx^1 \wedge \cdots \wedge dx^n \\
&= 0
\end{aligned}$$

Hence by definition, $\omega$ is closed. $\square$

(b) Let $\mathbb{S}^n = \{\vec{x} \in \mathbb{R}^{n+1} : |\vec{x}| = 1\}$. Given a smooth function $f : \mathbb{S}^n \to (0, \infty)$ and denote

$$\Sigma_f := \{f(\vec{x})\vec{x} : \vec{x} \in \mathbb{S}^n\}$$

i. Show that $\Sigma_f$ is an $n$-submanifold of $\mathbb{R}^{n+1} \setminus \{\vec{0}\}$.

**Solution. (1)** Since $f(\vec{x}) \neq 0$ for all $\vec{x} \in \mathbb{S}^n$, thus $\vec{0} \notin \Sigma_f$ and so $\Sigma_f$ is a subset of $\mathbb{R}^{n+1} \setminus \{\vec{0}\}$.

**(2)** Let $\vec{x} \in \mathbb{S}^n$ and let $F(u_1, \ldots, u_n) : \mathcal{U}_F \to \mathcal{O}_F$ be a local parametrization around $\vec{x}$. Define $\tilde{F} : \mathcal{U}_F \to \mathcal{V}_F$ by

$$\tilde{F}(u_1, \ldots, u_n) := f\big(F(u_1, \ldots, u_n)\big) F(u_1, \ldots, u_n)$$

where $\mathcal{V}_F := \{f(\vec{x})\vec{x} : \vec{x} \in \mathcal{O}_F\}$. The $\tilde{F}$ is a local parametrization around $f(\vec{x})\vec{x}$ on $\Sigma_f$.

Let $G(v_1, \ldots, v_n)$ be another local parametrization around $\vec{x}$ and let $\tilde{G}$ be the similarly induced local parametrization around $f(\vec{x})\vec{x}$ on $\Sigma_f$. Consider the transition map

$$(v_1, \ldots, v_n) = \tilde{G}^{-1} \circ \tilde{F}(u_1, \ldots, u_n)$$

Then we have $\tilde{G}(v_1, \dots, v_n) = \tilde{F}(u_1, \dots, u_n)$, i.e.

$$f\big(G(v_1, \dots, v_n)\big)G(v_1, \dots, v_n) = f\big(F(u_1, \dots, u_n)\big)F(u_1, \dots, u_n) \tag{2}$$

Since $F(u_1, \dots, u_n), G(v_1, \dots, v_n) \in \mathbb{S}^n$, thus by taking norm of both sides, and using the fact that $f > 0$, we obtain

$$f\big(G(v_1, \dots, v_n)\big) = f\big(F(u_1, \dots, u_n)\big)$$

Substitute this back into (2), we see that $G(v_1, \dots, v_n) = F(u_1, \dots, u_n)$. Thus

$$\tilde{G}^{-1} \circ \tilde{F}(u_1, \dots, u_n) = G^{-1} \circ F(u_1, \dots, u_n)$$

is smooth. Hence by definition, $\Sigma_f$ is a smooth $n$-manifold.

**(3)** Denote $\iota_f : \Sigma_f \to \mathbb{R}^{n+1} \setminus \{\vec{0}\}$ the inclusion map. Then we have

$$\iota_f \circ \tilde{F}(u_1, \dots, u_n) = f\big(F(u_1, \dots, u_n)\big)F(u_1, \dots, u_n)$$

By the product rule, we have

$$\frac{\partial \iota_f}{\partial u_i} = \frac{\partial (f \circ F)}{\partial u_i} F(u_1, \dots, u_n) + (f \circ F)(u_1, \dots, u_n)\frac{\partial F}{\partial u_i}$$

Here , since $F(u_1, \dots, u_n) \in \mathbb{R}^{n+1}$, it makes sense to speak of $\frac{\partial F}{\partial u_i}$ and it is tangent to $\mathbb{S}^n$. Thus, we identify $\frac{\partial F}{\partial u_i}$ as the tangent vector $\frac{\partial}{\partial u_i}$, as in the case of regular surface.

Taking $\mathcal{B} = \left\{\frac{\partial F}{\partial u_1}, \dots, \frac{\partial F}{\partial u_n}, F\right\}$ as a basis for $T_{f(\vec{x})\vec{x}}(\mathbb{R}^{n+1} \setminus \{\vec{0}\}) \simeq \mathbb{R}^{n+1}$, we see that

$$\left[\frac{\partial \iota_f}{\partial u_i}\right]_{\mathcal{B}} = \begin{pmatrix} 0 & \cdots & f \circ F & \cdots & 0 & \dfrac{\partial (f \circ F)}{\partial u_i} \end{pmatrix}^T$$

where $f \circ F$ is at the $i$-th position. Now, since the matrix

$$\begin{pmatrix} f \circ F & 0 & \cdots & 0 \\ 0 & f \circ F & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & f \circ F \\ \frac{\partial (f \circ F)}{\partial u_1} & \frac{\partial (f \circ F)}{\partial u_2} & \cdots & \frac{\partial (f \circ F)}{\partial u_n} \end{pmatrix} = \begin{pmatrix} (f \circ F) I_n \\ \\ \nabla (f \circ F) \end{pmatrix}$$

has full rank (because $f \neq 0$), this shows that the vectors $\frac{\partial \iota_f}{\partial u_i}$ are linearly independent. Thus, $[\iota_{f*}]$ has full rank. Therefore, $\iota_f$ is an immersion.

Hence by definition, $\Sigma_f$ is a submanifold of $\mathbb{R}^{n+1} \setminus \{\vec{0}\}$. $\square$

ii. Denote $\iota_f : \Sigma_f \to \mathbb{R}^{n+1} \setminus \{\vec{0}\}$ the inclusion map. Show that the absolute value of $\left|\int_{\Sigma_f} \iota_f^*\omega\right|$ is independent of the function $f : \mathbb{S}^n \to (0, \infty)$.

**Solution.** We need to show that for any smooth functions $f, g : \mathbb{S}^n \to (0.\infty)$, we have

$$\left|\int_{\Sigma_f} \iota_f^*\omega\right| = \left|\int_{\Sigma_g} \iota_g^*\omega\right| \tag{3}$$

Since $f, g$ are continuous and $\mathbb{S}^n$ is compact, both of them achieve a positive minimum on $\mathbb{S}^n$. Thus, there is an $r > 0$ such that $f, g > r$ on $\mathbb{S}^n$. Denote $\mathbb{S}^n(r)$ the sphere of radius $r$ centered at $\vec{0}$, and denote $\iota_r : \mathbb{S}^n(r) \to \mathbb{R}^{n+1} \setminus \{\vec{0}\}$ the inclusion map.

Let $M_f$ be the closed region bounded between $\Sigma_f$ and $\mathbb{S}^n(r)$. Then $M_f$ is a compact orientable smooth $(n+1)$-manifold with boundary

$$\partial M = \Sigma_f \sqcup \mathbb{S}^n(r)$$

This can be checked easily and we omit the details. Now, since by (a), $\omega$ is closed, thus by the generalized Stokes' Theorem,

$$0 = \int_M d\omega = \int_{\partial M} \omega = \varepsilon_f \int_{\Sigma_f} \iota_f^*\omega + \varepsilon_r \int_{\mathbb{S}^n(r)} \iota_r^*\omega$$

where $\varepsilon_f, \varepsilon_r \in \{\pm 1\}$ depends on the chosen orientation. Thus,

$$\varepsilon_f \int_{\Sigma_f} \iota_f^*\omega = -\varepsilon_r \int_{\mathbb{S}^n(r)} \iota_r^*\omega$$

Taking absolute value, we obtain

$$\left|\int_{\Sigma_f} \iota_f^*\omega\right| = \left|\int_{\mathbb{S}^n(r)} \iota_r^*\omega\right| \tag{4}$$

Now do the same for $g$; namely, consider $M_g$ the closed region bounded between $\Sigma_g$ and $\mathbb{S}^n(r)$. By the same argument, we also have

$$\left|\int_{\Sigma_g} \iota_g^*\omega\right| = \left|\int_{\mathbb{S}^n(r)} \iota_r^*\omega\right| \tag{5}$$

Combining (4) and (5) proves the desired equality (3). $\square$

iii. Find the value of the above integral. (Hint: It may be difficult to compute it directly, but you may pick a particular nice function $f$, and also find a nicer $n$-form $\eta$ on $\mathbb{R}^{n+1}$ such that $\iota_f^*\omega = \iota_f^*\eta$, then find the integral of $\iota_f^*\eta$ over $\Sigma_f$).

**Solution.** Note that the result of part ii also tells us that

$$\left|\int_{\Sigma_f} \iota_f^*\omega\right| = \left|\int_{\mathbb{S}^n(r)} \iota_r^*\omega\right| = \left|\int_{\mathbb{S}^n} \iota^*\omega\right|$$

where $\iota^* : \mathbb{S}^n \to \mathbb{R}^{n+1} \setminus \{\vec{0}\}$ denotes the inclusion map. Thus it suffices to compute the rightmost term.

Following the hint, we consider

$$\eta = \sum_{i=1}^{n+1} (-1)^i x_i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{n+1}$$

Clearly, $\eta$ is a smooth $n$-form defined on the whole $\mathbb{R}^{n+1}$. Moreover, since every point $\vec{x}$ on $\mathbb{S}^n$ satisfies $|\vec{x}| = 1$, thus $\iota_f^*\omega = \iota_f^*\eta$. Finally, by direct computation, we have

$$d\eta = (n+1)dx^1 \wedge \cdots \wedge dx^{n+1}$$

Therefore by the generalized Stokes' theorem again,

$$\begin{aligned}
\int_{\mathbb{S}^n} \iota^*\omega &= \int_{\mathbb{S}^n} \iota^*\eta \\
&= \int_{\partial B^{n+1}} \eta \\
&= \int_{B^{n+1}} d\eta \\
&= \int_{B^{n+1}} (n+1)dx^1 \wedge \cdots \wedge dx^{n+1} \\
&= (n+1)\int_{B^{n+1}} dx^1 \cdots dx^n \\
&= (n+1)\mathrm{Vol}(B^{n+1})
\end{aligned}$$

where $B^{n+1}$ is the closed unit ball in $\mathbb{R}^{n+1}$. Hence we conclude that

$$\left|\int_{\Sigma_f} \iota_f^*\omega\right| = (n+1)\mathrm{Vol}(B^{n+1}).$$

□