$0 \to G_1 \xrightarrow{T_1} G_2 \xrightarrow{T_2} G_3 \xrightarrow{T_3} \cdots \to G_k \xrightarrow{T_k} G_k \xrightarrow{T_k} 0.$

exact $\overset{\text{def}}{\iff} \ker T_k = \operatorname{Im} T_{k-1} \quad \forall k.$

$M = U \cup V$ open in M.

$U \cap V \xrightarrow{j_U} U$, $U \cap V \xrightarrow{j_V} V$, $U \xrightarrow{i_U} U \cup V$, $V \xrightarrow{i_V} U \cup V = M$.

S^1: U, V.

$U \cap V = \{ \}$

$\Phi : M \to N$

$\Phi^* : \Lambda^k N \to \Lambda^k M.$

$\Phi^* : H^k_{dR}(N) \to H^k_{dR}(M)$

$\Phi^*[\omega] := [\Phi^*\omega]$

$0 \to \Lambda^k(M) \xrightarrow{i_U^* \oplus i_V^*} \Lambda^k(U) \oplus \Lambda^k(V) \xrightarrow{j_U^* - j_V^*} \Lambda^k(U \cap V) \to 0$ is exact

$0 \to H^k_{dR}(M) \xrightarrow{i_U^* \oplus i_V^*} H^k_{dR}(U) \oplus H^k_{dR}(V) \xrightarrow{j_U^* - j_V^*} H^k_{dR}(U \cap V) \to 0$

$[\omega] \longmapsto ([i_U^*\omega], [i_V^*\omega])$

$([\omega], [\eta]) \longmapsto [i_U^*\omega - i_V^*\eta]$

Mayer–Vietoris sequence (Zigzag lemma)

$\exists \delta_k$

$0 \to H^0_{dR}(M) \xrightarrow{\Phi_0} H^0_{dR}(U) \oplus H^0_{dR}(V) \xrightarrow{\Psi_0} H^0_{dR}(U \cap V)$

$H^1_{dR}(M) \xrightarrow{\Phi_1} H^1_{dR}(U) \oplus H^1_{dR}(V) \xrightarrow{\Psi_1} H^1_{dR}(U \cap V)$

$H^2_{dR}(M) \xrightarrow{\Phi_2} H^2_{dR}(U) \oplus H^2_{dR}(V) \xrightarrow{\Psi_2} H^2_{dR}(U \cap V)$

$\xrightarrow{\delta_2} H^3_{dR}(M) \longrightarrow \cdots$

is exact.

e.g. $M = S^1 =$

$U \cap V = (\quad)$

$0 \to H^0_{dR}(O) \xrightarrow{\Phi_0} H^0_{dR}(\cap) \oplus H^0_{dR}(\cup) \xrightarrow{\Psi_0} H^0_{dR}(\ )$

$\mathbb{R} \quad \mathbb{R} \oplus \mathbb{R} \quad (1+1) \quad \mathbb{R}^2$

$0 = 1 - (1+1) + 2 - \chi \Rightarrow \chi = 1.$

$H^1_{dR}(O) \to H^1_{dR}(\cap) \oplus H^1_{dR}(\cup) \to \cdots$

Φ_0 surjective

$H^1_{dR}(\cap) \to$

$H^1_{dR}(S^1) = \operatorname{Im} \delta$

$\cong H^0_{dR}(\cap)/\ker \delta$

$= H^0_{dR}(\cap)/\operatorname{Im} \Psi_0$

$\cong \{(c_1, c_2) : c \in \mathbb{R}\}$

$\{(k_1 - k_2, k_1 - k_2) : k_i \in \mathbb{R}\}$

First iso thm

$G \xrightarrow{T} H$

$\operatorname{Im} T \cong G/\ker T.$

$Tg \leftarrow [g]$

$\dim H^1_{dR}(S^1) = 2 - 1 = 1.$

$S^1 \hookrightarrow \mathbb{R}^2$

$\alpha = i^*(-y\,dx + x\,dy)$ closed 1-form on S^1

$\int_{S^1} \alpha \neq 0 \Rightarrow$ not exact on S^1

$[\alpha] \neq [0]$ in $H^1_{dR}(S^1)$.

$\therefore H^1_{dR}(S^1) = \operatorname{Span}\{[\alpha]\}.$

$H^1_{dR}(S^1) \cong H^1_{dR}(\mathbb{R}^2 \setminus \{(0,0)\})$. S^1 def. retract of $\mathbb{R}^2 \setminus \{(0,0)\}$.

$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2}$ closed 1-form on $\mathbb{R}^2 \setminus \{(0,0)\}$

not exact. $\frac{-y^2 + x^2}{x^2 + y^2}$

$[\omega] \neq [0] \in H^1_{dR}(\mathbb{R}^2 \setminus \{(0,0)\})$

dimension $\Rightarrow H^1_{dR}(\mathbb{R}^2 \setminus \{(0,0)\}) = \operatorname{span}\{[\omega]\}.$

e.g. $S^2 =$ U $\cup$ V

$U \cong$

$U \cap V = \{ \}$ $\xrightarrow[\text{retract}]{\text{def}}$ S^1 $\quad H^1(S^1) \cong \mathbb{R}$

$0 \to H^0(S^2) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V)$

$\mathbb{R} \quad \mathbb{R} \oplus \mathbb{R} \quad \mathbb{R}$

$H^1(S^2) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V)$

$1 \quad 0 \quad 0 \quad 1$

$\mathbb{R}$

$\therefore$ injective surjective

$H^2(S^2) \to H^2(U) \oplus H^2(V) \to \cdots$

$0 \quad 0 \quad 1$

$\therefore H^2(S^2) \cong H^2(U \cap V)$

$\mathbb{R}$

$1 - 2 + 1 - \chi = 0$

$\Rightarrow \chi = 0.$

$\Rightarrow H(S^2) = 0.$

Useful result: Given $0 \to V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} V_3 \xrightarrow{T_3} \cdots \to V_k \xrightarrow{T_k} V_k \to 0$ is exact, and $\dim V_i < \infty$.

then $\dim V_1 - \dim V_2 + \dim V_3 - \cdots + (-1)^k \dim V_k = 0.$

$V \xrightarrow[\text{linear}]{T} W \Rightarrow \operatorname{Im}(T) \cong V/\ker T$

$\dim \operatorname{Im}(T) = \dim V - \dim \ker T$

$\dim V = \underbrace{\dim \operatorname{Im}(T)}_{\text{rank}(T)} + \underbrace{\dim \ker T}_{\text{nullity}(T)}.$

rank+nullity's Theorem

$0 \to V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} V_3 \xrightarrow{T_3} \cdots \to V_{k-1} \xrightarrow{T_{k-1}} V_k \to 0$

$\dim V_k = \dim \operatorname{Im}(T_{k-1}) = \dim V_{k-1} - \dim \ker T_{k-1}$

$= \dim V_{k-1} - \dim \operatorname{Im} T_{k-2}$ $\quad T_{k-2} : V_{k-2} \to V_k$

$= \dim V_{k-1} - (\dim V_{k-2} - \dim \ker T_{k-2})$

$= \dim V_{k-1} - \dim V_{k-2} + \dim \operatorname{Im} T_{k-3}$

$= \cdots$

$= \dim V_{k-1} - \dim V_{k-2} + \cdots (\pm) \dim \operatorname{Im} T_1$

$\mathbb{RP}^2 = (U_0 \cup U_1) \cup U_2$

$U_i = \{[x_0 : x_1 : x_2] : x_i \neq 0\} \cong \mathbb{R}^2$

$O_1 = U_0 \cup U_1$, $\quad U_0 = \{[1 : x : y]\}$ any real.

$U_0 \cap U_1 = \{[x_0 : x_1 : x_2]\} = \{[1 : y_1 : y_2] : y_2 \in \mathbb{R}, y_1 \neq 0\}$

$[1 : 0 : x_2] \to [1 : 1 : x]$

$[1 : t : x_2]$

$[0 : 1 : \tilde{x}]$

$[(1-t) : 1 : (1-t)x_2 + tx_2]$

y_1

$0 \to H^0(O_1) \to H^0(U_0) \oplus H^0(U_1) \to H^0(U_0 \cap U_1)$

$1 \quad (1+1) \quad 2$

$H^1(O_1) \to H^1(U_0) \oplus H^1(U_1) \to H^1(U_0 \cap U_1)$

$0 \quad 0$

$H^2(O_1) \to H^2(U_0) \oplus H^2(U_1) \to \cdots$

$0 \quad 0 \quad H^2(O_1) = 0.$

$1 - (1+1) + 2 - \chi = 0$

$\Rightarrow \chi = 1. \quad \therefore H^1(O_1) \cong \mathbb{R}.$

$\mathbb{RP}^2 = O_1 \cup U_2$

$U_0 \cup U_1$

$0 \to H^0(\mathbb{RP}^2) \to H^0(O_1) \oplus H^0(U_2) \to H^0(O_1 \cap U_2)$

$1 \quad (1+1) \quad 1$

$H^1(\mathbb{RP}^2) \to H^1(O_1) \oplus H^1(U_2) \to H^1(O_1 \cap U_2)$

$0 \quad (1+0) \quad 1$

$H^2(\mathbb{RP}^2) \to H^2(O_1) \oplus H^2(U_2) \to \cdots$

$0 \quad 0 \quad 0$

$y = 0$

$O_1 \cap U_2 = \{[x_0 : x_1 : x_2] : x_2 \neq 0 \text{ and } (x_0, x_1) \neq (0,0)\}$

$= \{[y_0 : y_1 : 1] : (y_0, y_1) \neq (0,0)\}$

$\cong \mathbb{R}^2 \setminus (0,0).$

S^1

$1 - 2 + 1 - x + 1 - 1 - y = 0$

$\Rightarrow x = y.$

$\dim H^0(O_1 \cap U_2) = 1$

$\dim H^1(O_1 \cap U_2) = \dim H^1(S^1) = 1.$

$\dim H^2(O_1 \cap U_2) = \dim H^2(S^1) = 0.$

M^n closed without boundary

$\dim H^n(M^n) = \begin{cases} 0 & \text{if } M^n \text{ is not orientable} \\ \mathbb{R} & \text{if } M^n \text{ is orientable.} \end{cases}$

$\mathbb{RP}^2$ $=$ $\cup$ $\mathbb{RP}^1$. $=$ U $\cup$ V

$\mathbb{RP}^1 \cong S^1.$

$U \cap V =$ def. retract S^1.

$0 \to H^0(\mathbb{RP}^2) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V)$

$1 \quad (1+1) \quad 1$

$H^1(\mathbb{RP}^2) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V)$

$x \quad (1+0) \quad 1$

$H^2(\mathbb{RP}^2) \to H^2(U) \oplus H^2(V) \to \cdots$

$y \quad (0+0)$

$1 - 2 + 1 - x + 1 - 1 + y = 0 \Rightarrow y = x.$

$\mathbb{RP}^3 =$ $S^3/\sim$ $=$ S^3_+ $\cup$ $\mathbb{RP}^2$.

S^2

$U \cong$ $\to$ $\mathbb{RP}^2$.

$= B^3.$

$=$

$b_k(M) = \dim H^k_{dR}(M)$

k-th Betti number.