MATH 4033 • Spring 2019 • Calculus on Manifolds
Problem Set #4 • Stokes' Theorem / Quotient Spaces • Due Date: 12/05/2019, 11:59PM

1. (a) Show that any complex manifold (i.e. transition maps are holomorphic) must be orientable. [Note: Please fix my calculation of determinant in class.]
   (b) A smooth manifold M^{2n} is called a symplectic manifold if there exists a smooth 2-form ω such that $d\omega = 0$, and for any $p \in M$, the only vector $X_p \in T_pM$ such that $i_{X_p}\omega_p = 0$ is the zero vector. Show that any symplectic manifold must be orientable.

2. Let M^n be a smooth manifold. For each $p \in M$ covered by local coordinates $(u_1, \cdots, u_n)$, we denote
$$\wedge^n T_p^* M := \text{span}\{du^1|_p \wedge \cdots \wedge du^n|_p\}.$$
Denote the n-form bundle of M by $\wedge^n T^* M := \bigcup_{p \in M} \{p\} \times \wedge^n T_p^* M$.
   (a) Show that the n-form bundle of M is a smooth manifold. What is its dimension?
   (b) Show that if M^n is orientable, then $\wedge^n T^* M$ is diffeomorphic to $M \times \mathbb{R}$.

3. Let ω be the n-form on $\mathbb{R}^{n+1} \backslash \{\vec{0}\}$ defined by:
$$\omega = \frac{1}{|\vec{x}|^{n+1}} \sum_{i=1}^{n+1} (-1)^{i-1} x_i \, dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{n+1}$$
where $\vec{x} = (x_1, \ldots, x_{n+1})$ and $|\vec{x}| = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$.
   (a) Show that ω is closed.
   (b) Let $\mathbb{S}^n = \{\vec{x} \in \mathbb{R}^{n+1} : |\vec{x}| = 1\}$. Given a smooth function $f : \mathbb{S}^n \to (0, \infty)$ and denote
$$\Sigma_f := \{f(\vec{x})\vec{x} : \vec{x} \in \mathbb{S}^n\}$$
      i. Show that Σ_f is an n-submanifold of $\mathbb{R}^{n+1} \backslash \{\vec{0}\}$.
      ii. Denote $\iota_f : \Sigma_f \to \mathbb{R}^{n+1} \backslash \{\vec{0}\}$ the inclusion map. Show that the absolute value of integral $\left| \int_{\Sigma_f} \iota_f^* \omega \right|$ is independent of the function $f : \mathbb{S}^n \to (0, \infty)$.
      iii. Find the value of the above integral. (Hint: It may be difficult to compute it directly, but you may pick a particular nice function f, and also find a nicer n-form η on $\mathbb{R}^{n+1}$ such that $\iota_f^* \omega = \iota_f^* \eta$, then find the integral of $\iota_f^* \eta$ over Σ_f).

4. Exercises 5.1 and 5.2

5. [Source: HKALE[1] - Year 1991, Pure Mathematics Paper 1 Q13]
Let $u \in \mathbb{R}^3$ be a unit vector. A relation $\sim$ on $\mathbb{R}^3$ is defined by:
$$v \sim w \text{ if and only if } v - w = ku \text{ for some } k \in \mathbb{R}.$$
   (a) Show that $\sim$ is an equivalence relation.
   (b) Denote by $[v]$ the equivalence class containing v. Let $f : \mathbb{R}^3/\sim \to \mathbb{R}^3$ be defined by $f([v]) = v - (v \cdot u)u$.
      i. Show that f is well-defined and injective.
      ii. For any non-zero $w \in \mathbb{R}^3$, show that w is perpendicular to u if and only if w is in the image of f. Hence deduce that f is not surjective.
   (c) Now given that $u = (0, 0, 1)$ and $v = (0, 1, 2)$, describe (or sketch) the set $[v]$.

[1] HKALE was known as the Hong Kong Advanced-Level Examination. It is the former local university entrance exam before 2012.