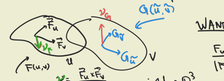
Orientable regular surface $\Sigma^2 \subset \mathbb{R}^3$

def $\exists\, C^\infty$ $\nu : \Sigma^2 \to \mathbb{R}^3$ s.t.

$|\nu| \equiv 1$ and $\nu(p) \perp T_p\Sigma^2$ $\forall p \in M$



WANT: $\dfrac{F_u \times F_v}{|F_u \times F_v|} = \dfrac{G_{\tilde u} \times G_{\tilde v}}{|G_{\tilde u} \times G_{\tilde v}|}$ for any overlapping F, G.

$F(u,v) = (x(u,v), y(u,v), z(u,v))$

$$\frac{\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}}{\left|\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}\right|} = \frac{\left(\det\frac{\partial(y,z)}{\partial(u,v)}\hat{i} + \det\frac{\partial(z,x)}{\partial(u,v)}\hat{j} + \det\frac{\partial(x,y)}{\partial(u,v)}\hat{k}\right)}{\left|\cdots\right|}$$

$$\frac{\frac{\partial G}{\partial \tilde u} \times \frac{\partial G}{\partial \tilde v}}{\left|\frac{\partial G}{\partial \tilde u} \times \frac{\partial G}{\partial \tilde v}\right|} = \frac{\left(\det\frac{\partial(y,z)}{\partial(\tilde u,\tilde v)}\hat{i} + \det\frac{\partial(z,x)}{\partial(\tilde u,\tilde v)}\hat{j} + \det\frac{\partial(x,y)}{\partial(\tilde u,\tilde v)}\hat{k}\right)}{\left|\cdots\right|}$$

$$\frac{\partial(y,z)}{\partial(u,v)} = \frac{\partial(y,z)}{\partial(\tilde u,\tilde v)}\frac{\partial(\tilde u,\tilde v)}{\partial(u,v)} \Rightarrow \det\frac{\partial(y,z)}{\partial(u,v)} \quad \det\frac{\partial(\tilde u,\tilde v)}{\partial(u,v)} \quad \det\frac{\partial(y,z)}{\partial(\tilde u,\tilde v)}$$

$$\det\frac{\partial(\tilde u,\tilde v)}{\partial(u,v)}\left(\det\frac{\partial(y,z)}{\partial(\tilde u,\tilde v)}\hat{i} + \det\frac{\partial(z,x)}{\partial(\tilde u,\tilde v)}\hat{j} + \det\frac{\partial(x,y)}{\partial(\tilde u,\tilde v)}\hat{k}\right)$$

$$\left|\det\frac{\partial(\tilde u,\tilde v)}{\partial(u,v)}\right|\left(\det\frac{\partial(y,z)}{\partial(\tilde u,\tilde v)}\hat{i} + \det\frac{\partial(z,x)}{\partial(\tilde u,\tilde v)}\hat{j} + \det\frac{\partial(x,y)}{\partial(\tilde u,\tilde v)}\hat{k}\right)$$

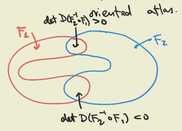
$$= \left(\det\frac{\partial(y,z)}{\partial(\tilde u,\tilde v)}\hat{i} + \det\frac{\partial(z,x)}{\partial(\tilde u,\tilde v)}\hat{j} + \det\frac{\partial(x,y)}{\partial(\tilde u,\tilde v)}\hat{k}\right)$$

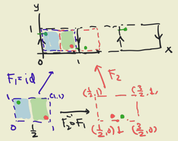
$$\Leftrightarrow \frac{\det\frac{\partial(\tilde u,\tilde v)}{\partial(u,v)}}{\left|\det\frac{\partial(\tilde u,\tilde v)}{\partial(u,v)}\right|} = 1 \Leftrightarrow \boxed{\det\frac{\partial(\tilde u,\tilde v)}{\partial(u,v)} > 0}$$

$\therefore \exists\, C^\infty$ $\nu : \Sigma^2 \to \mathbb{R}^3$, $|\nu| = 1$, $\nu_p \perp T_p\Sigma^2$ $\Leftrightarrow \exists$ atlas $\{F_\alpha : U_\alpha \to M\}$ s.t. $\det D(F_\beta^{-1} \circ F_\alpha) > 0$ $\forall \alpha, \beta$.

Orientable manifold M

def $\exists$ atlas $\{F_\alpha : U_\alpha \to M\}$ s.t. $\det D(F_\beta^{-1} \circ F_\alpha) > 0$ $\forall \alpha, \beta$.





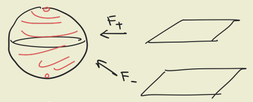
$$\Rightarrow \det D(F_2^{-1} \circ F_1)(x,y) = \begin{cases} -1 & \text{if } x \in (0, \frac{1}{2}) \\ 1 & \text{if } x \in (\frac{1}{2}, 1) \end{cases}$$

$(\mathbb{R} \times (0,1))/\sim$

$(x_1, y_1) \sim (x_2, y_2)$

def $\begin{cases} y_1 = y_2 & \text{if } x_2 - x_1 \text{ is even} \\ y_1 + y_2 = 1 & \text{if } x_2 - x_1 \text{ is odd} \end{cases}$

$$F_2^{-1} \circ F_1(x,y) = \begin{cases} (x+1, 1-y) & \text{if } x \in (0, \frac{1}{2}) \\ (x, y) & \text{if } x \in (\frac{1}{2}, 1) \end{cases}$$



$\det D(F_-^{-1} \circ F_+) < 0$.

$F_+(u,v)$, $G_+(u,v) = F_+(u,-v)$

$F_-(u,v)$, $G_-(\tilde u, \tilde v)$

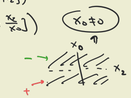
$\det D(G_-^{-1} \circ G_+) > 0$.

$\mathbb{RP}^2 = \{[x_0 : x_1 : x_2] \mid (x_0, x_1, x_2) \in \mathbb{R}^3 \setminus \{(0,0,0)\}\}$.

$F_0(x_1, x_2) := [1 : x_1 : x_2] : \mathbb{R}^2 \to \mathbb{RP}^2$.

$F_1(x_0, x_2) := [x_0 : 1 : x_2] : \mathbb{R}^2 \to \mathbb{RP}^2$.

$$F_0^{-1} \circ F_1(x_0, x_2) = F_0^{-1}([x_0 : 1 : x_2]) = F_0^{-1}\left(\left[1 : \tfrac{1}{x_0} : \tfrac{x_2}{x_0}\right]\right) = \left(\tfrac{1}{x_0}, \tfrac{x_2}{x_0}\right)$$



$$D(F_0^{-1} \circ F_1) = \frac{\partial\left(\frac{1}{x_0}, \frac{x_2}{x_0}\right)}{\partial(x_0, x_2)} = \begin{bmatrix} -\frac{1}{x_0^2} & 0 \\ 0 & \frac{1}{x_0} \end{bmatrix}$$

$\det D(F_1^{-1} \circ F_1) = -\frac{1}{x_0^2}$

e.g. T^*M of any C^∞ manifold M is orientable.

$F(u_1, \dots, u_n) \to M$.

$\tilde F(u_1, \dots, u_n, X_1, \dots, X_n) = \left(F(u_1, \dots, u_n), \sum_{i=1}^n X_i du^i\right)$

$\tilde G(v_1, \dots, v_n, Y_1, \dots, Y_n) = \left(G(v_1, \dots, v_n), \sum_j Y_j dv^j\right)$

$$\tilde G^{-1} \circ \tilde F = \left(G^{-1} \circ F(u_1, \dots, u_n), \sum_i X_i \frac{\partial u_i}{\partial v_1}, \dots, \sum_i X_i \frac{\partial u_i}{\partial v_n}\right)$$

$(u_1, \dots, u_n, X_1, \dots, X_n)$

$$D(\tilde G^{-1} \circ \tilde F) = \left[\begin{array}{c|c} D(G^{-1} \circ F) & 0 \\ \hline * & \frac{\partial u_i}{\partial v_j} \end{array}\right] \quad \left[\begin{array}{c|c} D(G^{-1} \circ F) & 0 \\ \hline * & D(F^{-1} \circ G) \end{array}\right]$$

$\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_n}, \frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}$

$$\det D(\tilde G^{-1} \circ \tilde F) = \det D(G^{-1} \circ F) \cdot \underbrace{\det D(F^{-1} \circ G)}_{= \frac{1}{\det D(G^{-1} \circ F)}} = 1 > 0.$$

e.g. complex manifolds are all orientable.

$G^{-1} \circ F(x_1, \dots, x_n, y_1, \dots, y_n) = (u_1, \dots, u_n, v_1, \dots, v_n)$

$(x_1, y_1) = x + iy$.

$$\begin{cases} \frac{\partial u_i}{\partial x_j} = \frac{\partial v_i}{\partial y_j} \\ \frac{\partial v_i}{\partial x_j} = -\frac{\partial u_i}{\partial y_j} \end{cases} \quad D(G^{-1} \circ F) = \begin{array}{c} \\ u \\ v \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{c|c} \frac{\partial u_i}{\partial x_j} & \frac{\partial u_i}{\partial y_j} \\ \hline \frac{\partial v_i}{\partial x_j} & \frac{\partial v_i}{\partial y_j} \end{array}\right] \end{array}$$

$$= \left[\begin{array}{c|c} \frac{\partial u_i}{\partial x_j} & \frac{\partial u_i}{\partial y_j} \\ \hline -\frac{\partial u_i}{\partial y_j} & \frac{\partial u_i}{\partial y_j} \end{array}\right]$$

$$0 \neq \det D(G^{-1} \circ F) = \left(\det\frac{\partial(u_1, \dots, u_n)}{\partial(x_1, \dots, x_n)}\right)^2 + \left(\det\frac{\partial(u_1, \dots, u_n)}{\partial(y_1, \dots, y_n)}\right)^2 > 0$$

e.g. Symplectic manifold (M^{2n}, ω), $d\omega = 0$, ω 2-form, ω^n

$\{\omega(X, Y) = 0 \ \forall Y \Rightarrow X = 0\}$

$\{i_X \omega = 0 \Rightarrow X = 0\}$

orientable $M^n \Leftrightarrow \exists\, C^\infty$ n-form Ω s.t. $\Omega(p) \neq 0$ $\forall p \in M$.

$\mathbb{Z}/7\mathbb{Z} = \{[m] : m \in \mathbb{Z}\} = \{m + 7\mathbb{Z} : m \in \mathbb{Z}\}$

$[m] = [n] \Leftrightarrow m - n \in 7\mathbb{Z}$.

$$= \begin{cases} m + 7\mathbb{Z} = \{m + 7k : k \in \mathbb{Z}\} \\ n + 7\mathbb{Z} = \{n + 7\ell : \ell \in \mathbb{Z}\} \end{cases}$$

$\Leftrightarrow m - n \in 7\mathbb{Z}$.

$\int_M f\, du^1 \wedge \dots \wedge du^n$ "$=$" informal $\int_M f\, du^1 \dots du^n$.

$\int_{\mathbb{R}^2} f\, dx \wedge dy = \int_{\mathbb{R}^2} f\, dx\, dy$ $\neq$ $\int_{\mathbb{R}^2} -f\, dy \wedge dx = -\int_{\mathbb{R}^2} f\, dy\, dx$

$\boxed{\int f\, dx\, dy = \int f\, dy\, dx}$

Fix $(u_1, \dots, u_n) \leftarrow$ order

$\int_{M^n} f\, du^1 \wedge \dots \wedge du^n := \int_M f\, du^1 \dots du^n$.

are $\to$ M^n client

$\Omega = g\, du^2 \wedge du^1 \wedge du^3 \wedge \dots \wedge du^n$

$\int_{M^n} \Omega = \int_M (-g)\, du^1 \wedge du^2 \wedge \dots \wedge du^n = -\int_M g\, du^1 \dots du^n$.

$(x_1, \dots, x_n)$

$\int_M f\, dx^1 \wedge \dots \wedge dx^n \overset{?}{=} \int_M f\, du^1 \wedge \dots \wedge du^n$.

$f\, dx^1 \wedge \dots \wedge dx^n = f \det\frac{\partial(x_1, \dots, x_n)}{\partial(u_1, \dots, u_n)} du^1 \wedge \dots \wedge du^n$.

$\int_M f\, dx^1 \wedge \dots \wedge dx^n = \int_M f \det\frac{\partial(x_1, \dots, x_n)}{\partial(u_1, \dots, u_n)} du^1 \wedge \dots \wedge du^n$.

WANT $\| = \int_M f \det\frac{\partial(x_1, \dots, x_n)}{\partial(u_1, \dots, u_n)} du^1 \dots du^n$

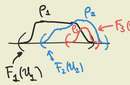
$\int_M f\, dx^1 \dots dx^n = \int_M f \left|\det\frac{\partial(x_1, \dots, x_n)}{\partial(u_1, \dots, u_n)}\right| du^1 \dots du^n$

[Need: M is orientable.]

$M \xrightarrow{F_\alpha} U_\alpha$ $\{F_\alpha : U_\alpha \to M\}$ oriented.

$\exists \begin{cases} \forall \alpha, \\ \rho_\alpha : M \to [0,1] \ C^\infty \end{cases}$

- $\rho_\alpha \equiv 0$ outside $F_\alpha(U_\alpha)$.
- $\sum_\alpha \rho_\alpha \equiv 1$ on M.



$$\Omega = \left(\sum_\alpha \rho_\alpha\right)\Omega = \sum_\alpha (\rho_\alpha \Omega)$$

$\int_{F_\alpha(U_\alpha)} \rho_\alpha \Omega$ ✓

$\int_M \Omega := \sum_\alpha \int_{F_\alpha(U_\alpha)} \rho_\alpha \Omega$

Well-definedness?

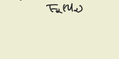
$\{F_\alpha : U_\alpha \to M\}$ oriented

$\{G_\beta : V_\beta \to M\}$.

$\phi_\beta : M \to [0,1]$

$\operatorname{supp}(\phi_\beta) \subset G_\beta(V_\beta)$

- $\sum_\beta \phi_\beta \equiv 1$ ←



$\rho_\alpha : M \to [0,1]$ smooth.

$\operatorname{supp}(\rho_\alpha) \subset F_\alpha(U_\alpha)$.

- $\sum_\alpha \rho_\alpha \equiv 1$.

$$\sum_\alpha \int_{F_\alpha(U_\alpha)} \rho_\alpha \Omega \overset{\text{WANT}}{=} \sum_\beta \int_{G_\beta(V_\beta)} \phi_\beta \Omega.$$

Proof:

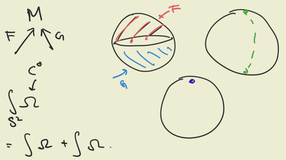
$$\sum_\alpha \int_{F_\alpha(U_\alpha)} \rho_\alpha \Omega = \sum_\alpha \int_{F_\alpha(U_\alpha)} \underbrace{\left(\sum_\beta \phi_\beta\right)}_{\equiv 1} \rho_\alpha \Omega$$

$$= \sum_{\alpha,\beta} \int_{F_\alpha(U_\alpha)} \rho_\alpha \phi_\beta \Omega$$

$$= \sum_{\alpha,\beta} \int_{F_\alpha(U_\alpha) \cap G_\beta(V_\beta)} \rho_\alpha \phi_\beta \Omega$$

$$\sum_\beta \int_{G_\beta(V_\beta)} \phi_\beta \Omega = \sum_\beta \int_{G_\beta(V_\beta)} \left(\sum_\alpha \rho_\alpha\right) \phi_\beta \Omega.$$

$\sum_\alpha \rho_\alpha = 1$



$= \int_{F(U)} \Omega + \int_{G(V)} \Omega$.

- $\int_{F_\alpha(U_\alpha)} f\, du_\alpha^1 \wedge \dots \wedge du_\alpha^n = \int_{F_\alpha(U_\alpha)} f\, du_\alpha^1 \dots du_\alpha^n$.
- $\int_M \Omega = \sum_\alpha \int_{F_\alpha(U_\alpha)} \rho_\alpha \Omega$

- $\rho_\alpha : M \to [0,1]$, C^∞. $\operatorname{supp} \rho_\alpha \subset F_\alpha(U_\alpha)$.
- $\sum_\alpha \rho_\alpha \equiv 1$ on M.

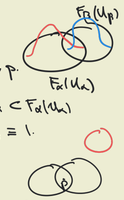
M^n orientable $\Leftrightarrow \exists\, C^\infty$ (global) n-form Ω on M s.t. $\Omega(p) \neq 0$ $\forall p \in M$.

$(\Rightarrow)$ $\mathcal{A} = \{F_\alpha(u_\alpha^1, \dots, u_\alpha^n) : U_\alpha \to M\}$ oriented atlas.

$\Omega = \sum_\alpha \rho_\alpha\, du_\alpha^1 \wedge \dots \wedge du_\alpha^n$

$p \in M$, fix $(v_1, \dots, v_n)$ local coordinates near p. $\in \mathcal{A}$.

$$\Omega_p = \sum_\alpha \rho_\alpha \det\frac{\partial(u_\alpha^1, \dots, u_\alpha^n)}{\partial(v_1, \dots, v_n)} dv^1 \wedge \dots \wedge dv^n \Big| \begin{array}{l} \operatorname{supp} \rho_\alpha \subset F_\alpha(U_\alpha) \\ \sum \rho_\alpha \equiv 1. \end{array}$$



$$= \left(\sum_\alpha \rho_\alpha \det\frac{\partial(u_\alpha^1, \dots, u_\alpha^n)}{\partial(v_1, \dots, v_n)}\right) dv^1 \wedge \dots \wedge dv^n$$

$\in [0,1]$ $+$

$\sum_\alpha \rho_\alpha(p) = 1 \Rightarrow \exists \alpha$ s.t. $\rho_\alpha(p) > 0$.

$(\Leftarrow)$ $\exists\, \Omega$ C^∞ n-form on M, $\Omega(p) \neq 0$ $\forall p \in M$.

$$\text{A. } \begin{cases} \Omega\left(\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_n}\right) > 0 & ① \\ \Omega\left(\frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial v_1}\right) > 0 & ② \end{cases} \Rightarrow \det\frac{\partial(v_1, \dots, v_n)}{\partial(u_1, \dots, u_n)} > 0$$

let $\Omega = f\, du^1 \wedge \dots \wedge du^n$

① $\Rightarrow f > 0$.

② $\Rightarrow f\, du^1 \wedge \dots \wedge du^n \left(\frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial v_n}\right) > 0$

$\Rightarrow f \det\frac{\partial(u_1, \dots, u_n)}{\partial(v_1, \dots, v_n)} > 0$.

let $\mathcal{A}' = \{F_\alpha(u_1^\alpha, \dots, u_n^\alpha) : U_\alpha \to M\}$

For each α, define $G_\alpha(v_1^\alpha, \dots, v_n^\alpha) = \begin{cases} F_\alpha(u_1^\alpha, \dots, u_n^\alpha) & \text{if } \Omega\left(\frac{\partial}{\partial u_1^\alpha}, \dots, \frac{\partial}{\partial u_n^\alpha}\right) > 0 \\ F_\alpha(-u_1^\alpha, \dots, v_n^\alpha) & \text{if } \dots < 0 \end{cases}$

$\Rightarrow \Omega\left(\frac{\partial}{\partial v_1^\alpha}, \dots, \frac{\partial}{\partial v_n^\alpha}\right) > 0$ $\forall \alpha$.

$\det\frac{\partial(v_1^\alpha, \dots, v_n^\alpha)}{\partial(v_1^\beta, \dots, v_n^\beta)} > 0$. $\forall \alpha, \beta$ $\Rightarrow \mathcal{A} = \{G_\alpha(v_1^\alpha, \dots, v_n^\alpha) : U_\alpha \to M\}$ oriented atlas.

M^n orientable manifold.

$\partial M \neq \emptyset$ or $= \emptyset$.

Stokes' Theorem:

$$\int_{\partial M} \omega = \int_M d\omega$$

$(n-1)$-form, n-form

① Choose Ω to be the orientation of M.

$\Downarrow$

② Declare $i_\eta \Omega$ to be the orientation of ∂M.

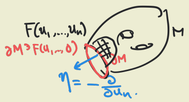
$(u_1, \dots, u_n)$

$\Omega\left(\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_n}\right) > 0$.

$\uparrow$ choose

$\int_{F_\alpha(U_\alpha)} f\, du_\alpha^1 \wedge \dots \wedge du_\alpha^n$

$$= \begin{cases} \int_{F_\alpha(U_\alpha)} f\, du_\alpha^1 \dots du_\alpha^n & \text{if } \Omega\left(\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_n}\right) > 0 \\ -\int_{F_\alpha(U_\alpha)} f\, du_\alpha^1 \dots du_\alpha^n & \text{if } \dots < 0. \end{cases}$$



If $\Omega\left(\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_n}\right) > 0$.

$$\Rightarrow (i_\eta \Omega)\left(\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_{n-1}}\right)$$

$$= \Omega\left(\eta, \frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_{n-1}}\right)$$

$$= \Omega\left(-\frac{\partial}{\partial u_n}, \frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_{n-1}}\right)$$

$$= (-1)^n \Omega\left(\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_n}\right)$$

$\int_M f\, du^1 \wedge \dots \wedge du^n = \int f\, du^1 \dots du^n$

Want: $\int_{\partial M} \varphi\, du^1 \wedge \dots \wedge du^{n-1} = (-1)^n \int_{\partial M} \varphi\, du^1 \dots du^{n-1}$.

Proof of Stokes'

Step ① Assume

$$\omega = \sum_i \omega_i\, du^1 \wedge \dots \wedge \widehat{du^i} \wedge \dots \wedge du^n$$

$$d\omega = \sum_{i,j} \frac{\partial \omega_i}{\partial u_j} du^j \wedge du^1 \wedge \dots \wedge \widehat{du^i} \wedge \dots \wedge du^n.$$

$$= \sum_i \frac{\partial \omega_i}{\partial u_i} (-1)^{i-1} du^1 \wedge \dots \wedge du^n$$

WANT: $\int_M d\omega = 0$.

$$\int_M d\omega = \int_{F(U)} \sum_i \frac{\partial \omega_i}{\partial u_i} (-1)^i\, du^1 \wedge \dots \wedge du^n.$$

$$= \sum_i \int_{F(U)} \frac{\partial \omega_i}{\partial u_i} (-1)^i\, du^1 \dots du^n.$$

$$= \sum_i \int_{-R}^{R} \cdots \int_{-R}^{R} \frac{\partial \omega_i}{\partial u_i} (-1)^i\, du^1 du^2 \dots du^n$$

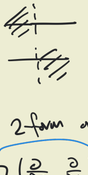
$$= \sum_i \int_{-R}^{R} \cdots \int_{-R}^{R} \left[(-1)^i \omega_i \Big|_{u_i = -R}^{u_i = R}\right] du^1 \dots \widehat{du^i} \dots du^n$$

$= 0$.

$\mathbb{RP}^2$

$F_0 \uparrow F_1 \uparrow F_2$

$$\det D(F_1^{-1} \circ F_0) \begin{cases} > 0 \\ < 0 \end{cases}$$



Assume $\mathbb{RP}^2$ orientable

$\Rightarrow \exists \Omega$ non-vanishing global 2-form on $\mathbb{RP}^2$.

$F_0(u_1, u_2)$

$F_1(v_0, v_2)$

$$\Omega\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right) \cdot \Omega\left(\frac{\partial}{\partial v_0}, \frac{\partial}{\partial v_2}\right) > 0 \text{ on } W_+$$

$$\Omega\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right) \cdot \Omega\left(\frac{\partial}{\partial v_0}, \frac{\partial}{\partial v_2}\right) < 0 \text{ on } W_-$$

WLOG

$\Rightarrow$ assume $\Omega\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right) > 0$ on W_+, $\Omega\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right) < 0$ on W_-.

$\Rightarrow \exists p$ s.t. $\Omega\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right) = 0$ ⊗.