MATH 4033 • Calculus on Manifolds

Spring 2019, HKUST

Solutions to Exercises in Lecture Notes

Disclaimer: The solutions are provided by former students of MATH 4033. The correctness are not absolutely guaranteed. You are welcomed to discuss them with the lecturer or the TA if you have any doubt.

Chapter 1

Exercise 1.1.

(1) Since $\sin x$, $\sin 2x$ and x are C^k $\forall k$, F is smooth.

(3) $\frac{\partial F}{\partial u} = (\cos u, 2\cos 2u, 0)$

$\frac{\partial F}{\partial v} = (0, 0, 1)$

$\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} \neq \mathbf{0}$

(2) $\forall v_n = v \in (0,1)$, let $u_n = 2\pi - \frac{1}{n}$. Now, $\{(u_n, v_n)\}$ is trivially a diverging sequence according to the domain $(0, 2\pi) \times (0,1)$. But $\lim_{n\to+\infty} F(u_n, v_n) = (0, 0, v)$, which means $F(u_n, v_n)$ is convergent. Hence, F^{-1} maps convergent sequence $F(u_n, v_n)$ to divergent sequence $\{(u_n, v_n)\}$. By theorem, F^{-1} is not continuous and F is not a homeomorphism.

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Exercise 1.2.

- Left: $F_3(u,v) = (-\sqrt{1-u^2-v^2}, v, u) : B_1(0) \to S^2_{left} = \{x^2+y^2+z^2 = 1 | x < 0\}$
- Right: $F_4(u,v) = (\sqrt{1-u^2-v^2}, v, u) : B_1(0) \to S^2_{right} = \{x^2+y^2+z^2 = 1 | x > 0\}$
- Front: $F_5(u,v) = (u, -\sqrt{1-u^2-v^2}, v) : B_1(0) \to S^2_{front} = \{x^2+y^2+z^2 = 1 | y < 0\}$
- Back: $F_6(u,v) = (u, \sqrt{1-u^2-v^2}, v) : B_1(0) \to S^2_{back} = \{x^2+y^2+z^2 = 1 | y > 0\}$

■

Exercise 1.3.
Since $F_+(u,v) = (\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1})$, thus

$$\frac{\partial F_+}{\partial u} = (\frac{2(1-u^2+v^2)}{(u^2+v^2+1)^2}, -\frac{4uv}{(u^2+v^2+1)^2}, \frac{4u}{(u^2+v^2+1)^2})$$

$$\frac{\partial F_+}{\partial v} = (-\frac{4uv}{(u^2+v^2+1)^2}, \frac{2(1+u^2-v^2)}{(u^2+v^2+1)^2}, \frac{4v}{(u^2+v^2+1)^2})$$

$$\therefore \quad \frac{\partial F_+}{\partial u} \times \frac{\partial F_+}{\partial v} = (-\frac{8u}{(u^2+v^2+1)^3}, -\frac{8v}{(u^2+v^2+1)^3}, -\frac{4(u^2+v^2-1)}{(u^2+v^2+1)^3})$$

If $\frac{\partial F_+}{\partial u} \times \frac{\partial F_+}{\partial v} = 0$, then it is necessary that

$$\begin{cases} u = 0 \\ v = 0 \\ u^2 + v^2 - 1 = 0 \end{cases}$$

which is impossible.
$\therefore$ Condition (3) is satisfied.

To find F_+^{-1}, we set $(x, y, z) = F_+(u, v)$ and solve for (u, v) in terms of x, y, z.

$$\begin{cases} x = \frac{2u}{u^2+v^2+1} \\ y = \frac{2v}{u^2+v^2+1} \\ z = \frac{u^2+v^2-1}{u^2+v^2+1} \end{cases} \implies \begin{cases} x = u \cdot \frac{2}{u^2+v^2+1} \\ y = v \cdot \frac{2}{u^2+v^2+1} \\ 1 - z = \frac{2}{u^2+v^2+1} \end{cases} \implies \begin{cases} x = u(1-z) \\ y = v(1-z) \end{cases} \implies \begin{cases} u = \frac{x}{1-z} \\ v = \frac{y}{1-z} \end{cases}$$

$$\therefore \quad F_+^{-1}(x, y, z) = (\frac{x}{1-z}, \frac{y}{1-z})$$

■

Exercise 1.4.

Denote E be the ellipsoid. Let $F_+(u, v) : \mathbb{R}^2 \to E \setminus \{(0, 0, c)\}$ and $F_-(u, v) : \mathbb{R}^2 \to E \setminus \{(0, 0, -c)\}$.

For F_+, we let $F_+(u, v) = (\xi, \eta, \zeta)$. Then similar to stereographic parametrization of sphere in notes, assign each point $(u, v, 0)$ on the xy-plane of $\mathbb{R}^3$ to a point where the line segment joining $(u, v, 0)$ and the north pole $(0, 0, c)$ intersects the ellipsoid. So we consider

$$\begin{cases} \frac{v}{u} = \frac{\eta}{\xi} \\ \frac{u}{-c} = \frac{\xi}{\zeta - c} \\ \frac{v}{-c} = \frac{\eta}{\zeta - c} \end{cases}$$

These implies

$$\xi = \frac{2a^2b^2u}{b^2u^2 + a^2v^2 + a^2b^2}, \quad \eta = \frac{2a^2b^2v}{b^2u^2 + a^2v^2 + a^2b^2}, \quad \zeta = \frac{(b^2u^2 + a^2v^2 - a^2b^2)c}{b^2u^2 + a^2v^2 + a^2b^2}$$

Hence

$$F_+(u, v) = (\frac{2a^2b^2u}{b^2u^2 + a^2v^2 + a^2b^2}, \frac{2a^2b^2v}{b^2u^2 + a^2v^2 + a^2b^2}, \frac{(b^2u^2 + a^2v^2 - a^2b^2)c}{b^2u^2 + a^2v^2 + a^2b^2}).$$

Since a, b are positive constants, both F_+ is trivially smooth. It is surjective according to the range $E \setminus \{(0, 0, c)\}$. Note that $\frac{(b^2u^2+a^2v^2-a^2b^2)c}{b^2u^2+a^2v^2+a^2b^2} = 1 - \frac{2a^2b^2c}{b^2u^2+a^2v^2+a^2b^2}$. If $\exists u_1, u_2, v_1, v_2$,

$$\begin{cases} \frac{2a^2b^2u_1}{b^2u_1^2+a^2v_1^2+a^2b^2} = \frac{2a^2b^2u_2}{b^2u_2^2+a^2v_2^2+a^2b^2} \\ \frac{2a^2b^2v_1}{b^2u_1^2+a^2v_1^2+a^2b^2} = \frac{2a^2b^2v_2}{b^2u_2^2+a^2v_2^2+a^2b^2} \\ 1 - \frac{2a^2b^2c}{b^2u_1^2+a^2v_1^2+a^2b^2} = 1 - \frac{2a^2b^2c}{b^2u_2^2+a^2v_2^2+a^2b^2} \end{cases} \implies b^2u_1^2 + a^2v_1^2 + a^2b^2 = b^2u_2^2 + a^2v_2^2 + a^2b^2$$

$$\implies \begin{cases} u_1 = u_2 \\ v_1 = v_2 \end{cases}$$

Hence F_+ is 1-1. Again using the stereographic parametrization of ellipzoid, we find that $F_+^{-1}(x,y,z) = (\frac{-cx}{z-c}, \frac{-cy}{z-c})$ and both F_+ and F_+^{-1} are continuous. Finally,

$$\frac{\partial F_+}{\partial u} = (\frac{2a^2b^2(-b^2u^2+a^2v^2+a^2b^2)}{(b^2u^2+a^2v^2+a^2b^2)^2}, \frac{-4a^2b^4uv}{(b^2u^2+a^2v^2+a^2b^2)^2}, \frac{4a^2b^4cu}{(b^2u^2+a^2v^2+a^2b^2)^2})$$
$$\frac{\partial F_+}{\partial v} = (\frac{-4a^4b^2uv}{(b^2u^2+a^2v^2+a^2b^2)^2}, \frac{2a^2b^2(b^2u^2-a^2v^2+a^2b^2)}{(b^2u^2+a^2v^2+a^2b^2)^2}, \frac{4a^4b^2cv}{(b^2u^2+a^2v^2+a^2b^2)^2})$$

and $\frac{\partial F_+}{\partial u} \times \frac{\partial F_+}{\partial v} \neq 0$. Hence F_+ is a smooth local parametrization. Similarly,

$$F_-(u,v) = (\frac{2a^2b^2u}{b^2u^2+a^2v^2+a^2b^2}, \frac{2a^2b^2v}{b^2u^2+a^2v^2+a^2b^2}, -\frac{(b^2u^2+a^2v^2-a^2b^2)c}{b^2u^2+a^2v^2+a^2b^2})$$

is another smooth local parametrization. ■

Exercise 1.5.

(a) $\begin{cases} \frac{y(u)-1}{x(u)} = \frac{1}{-u} \\ x(u)^2 + y(u)^2 = 1 \end{cases} \implies y(u) = -\frac{x(u)}{u} + 1$

$\implies (1+\frac{1}{u^2})x(u)^2 - \frac{2}{u}x(u) = 0$

$\implies \begin{cases} x(u) = \frac{2u}{u^2+1} \\ y(u) = \frac{u^2-1}{u^2+1}. \end{cases}$

(b) Image of $F_1 = \{(x,y,z) \in \mathbb{R}^3 \setminus \{(0,1,h)|\forall h\} | x^2+y^2=1\}$.

(c) $F_1 = (\frac{2u}{u^2+1}, \frac{u^2-1}{u^2+1}, v)$ which is smooth and continuous.

$(\frac{2u}{u^2+1})^2 + (\frac{u^2-1}{u^2+1})^2 = 1$ so $(\frac{2u}{u^2+1}, \frac{u^2-1}{u^2+1})$ maps u to every point on the unit circle $\forall z = v \in \mathbb{R}$ and F_1 is surjective.

If $F_1(u_1,v_1) = F_1(u_2,v_2)$, $\begin{cases} \frac{2u_1}{u_1^2+1} &= \frac{2u_2}{u_2^2+1} \\ \frac{u_1^2-1}{u_1^2+1} &= \frac{u_2^2-1}{u_2^2+1} \end{cases} \implies \begin{cases} u_1 &= u_2 \\ v_1 &= v_2 \end{cases}$. Thus F_1 is 1-1.

Next, consider $\begin{cases} x = \frac{2u}{u^2+1} \\ y = \frac{u^2-1}{u^2+1} \\ z = v \end{cases}$. We have $F_1^{-1}(x,y,z) = (\frac{x}{1-y}, z) \forall y \neq 1$ which is continuous on O_1.

$\frac{\partial F_1}{\partial u} \times \frac{\partial F_1}{\partial v} = (\frac{2-2u^2}{(1+u^2)^2}, \frac{4u}{(1+u^2)^2}, 0) \times (0,0,1) = (\frac{4u}{1+u^2}^2, \frac{2u^2-2}{(1+u^2)^2}, 0) \neq 0 \forall (u,v) \in \mathbb{R}^2$. In short, F_1 is smooth.

(d) $F_2 = (\frac{2u}{u^2+1}, \frac{1-u^2}{1+u^2}, v) \forall (x,y,z) \in \mathbb{R}^3 \setminus (\xi, -1, \zeta)$

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Exercise 1.6.

$f(x,y,z) = (x+y+z-1)^2 = c$ is a level surface of $f^{-1}(c)$. Then

$$\begin{aligned} \nabla f(x,y,z) &= [\frac{\partial f(x,y,z)}{\partial x}, \frac{\partial f(x,y,z)}{\partial y}, \frac{\partial f(x,y,z)}{\partial z}] \\ &= [2(x+y+z-1)(y+z), 2(x+y+z-1)(x+z), 2(x+y+z-1)(y+z)] \\ &= 2(x+y+z-1)[y+z, x+z, y+z] \\ &= \mathbf{0} \quad \text{iff} \quad x+y+z=1 \quad \text{or} \quad \begin{cases} x = -y \\ y = -z \\ x = -z \end{cases} \end{aligned}$$

- Case I: if $\exists p = (x_0, y_0, z_0)$ such that $x_0 + y_0 + z_0 = 1$ and $c = f(x_0, y_0, z_0) = 0$, then $f^{-1}(0)$ is non-empty level surface.
  Consider $c = 0$,

$$\begin{aligned} f(x,y,z) = (x+y+z-1)^2 = 0 &\Rightarrow x+y+z-1=0 \\ &\Rightarrow x+y+z=1 \\ &\Rightarrow f^{-1}(0) = \{(x,y,z)|x+y+z=1\} \end{aligned}$$

  $f^{-1}(0)$ is a plane in 3D which can be parametrized locally by $F(x,y) = (x, y, 1-x-y)$. Thus $f^{-1}(0)$ is a regular surface even though $\nabla f(x,y,z) = 0$

- Case II: $\begin{cases} x = -y \\ y = -z \\ x = -z \end{cases}$
  The only case for satisfying the equations is $x = y = z = 0$.
  $c = f(0,0,0) = (0+0+0-1)^2 = 1$. Then $f^{-1}(1)$ is a non-empty level surface. As $\nabla f(0,0,0) = \mathbf{0}$, and $(0,0,0) \in f^{-1}(1)$, thus Theorem 1.6 fails to give any conclusion that $f^{-1}(1)$ is a regular surface.
  However, As

$$\begin{aligned} f^{-1}(1) &= \{(x,y,z)|x+y+z=0 \text{ or } x+y+z=2\} \\ &= \{(x,y,z)|x+y+z=0\} \cup \{(x,y,z)|x+y+z=2\} \end{aligned}$$

  both $x+y+z=0$ and $x+y+z=2$ are planes in 3D as both can be parametrized as $F(x,y) = (x, y, -x-y)$ and $F(x,y) = (x, y, 2-x-y)$ respectively.
  So $f^{-1}(1)$ is also a regular surface

- Case III: for $c \neq 0, 1$,

$$\begin{aligned} &f(x,y,z) = (x+y+z-1)^2 = c \\ &\Rightarrow x+y+z = 1 \pm \sqrt{c} \neq 0 \\ &\Rightarrow f^{-1}(c) = \{(x,y,z)|x+y+z = 1+\sqrt{c} \text{ or } x+y+z = 1-\sqrt{c}\} \end{aligned}$$

  $\therefore \forall (x,y,z) \in f^{-1}(c), \nabla f(x,y,z) \neq \mathbf{0}$. By theorem 1.6, $f^{-1}(c)$ is a regular surface.

Therefore, for any c, $f^{-1}(c)$ is a regular surface. ■

Exercise 1.7.

Let $g(x,y,z) = R^2 - (\sqrt{x^2+y^2} - r)^2 - z^2 = 0$. Then $g^{-1}(0)$ is a level surface. Then

$$\begin{aligned} \nabla g(x,y,z) &= [\frac{\partial g(x,y,z)}{\partial x}, \frac{\partial g(x,y,z)}{\partial y}, \frac{\partial g(x,y,z)}{\partial z}] \\ &= [\frac{-2x(\sqrt{x^2+y^2}-r)}{\sqrt{x^2+y^2}}, \frac{-2y(\sqrt{x^2+y^2}-r)}{\sqrt{x^2+y^2}}, -2z] \end{aligned}$$

$\nabla g(x,y,z) = \mathbf{0}$ only when $z = 0$ and $\sqrt{x^2+y^2} = r \Rightarrow x^2+y^2 = r^2$.
As given $r > 0$, $x^2+y^2 = r^2 \neq 0 \Rightarrow$ at least one of x or y is not zero.
But when $x^2+y^2 = r^2, z = 0$, $g(x,y,0) = R^2 \neq 0$. As given $R > r > 0 \Rightarrow R^2 > r^2 > 0 \Rightarrow (x,y,0) \notin g^{-1}(0)$ given $x^2+y^2 = r^2$.
Hence $g^{-1}(0)$ is a regular surface as $\forall (x,y,z) \in g^{-1}(0), \nabla g(x,y,z) \neq \mathbf{0}$. ■

Exercise 1.8.

Given any point $p \in g^{-1}(c)$, we assume $\frac{\partial g}{\partial y}(p) \neq 0$.

Since $\frac{\partial F}{\partial u}(p) \times \frac{\partial F}{\partial v}(p) \neq 0$ is parallel to $\nabla g(p)$, then by writing $F(u,v) = (x(u,v), y(u,v), z(u,v))$, we have

$$\begin{aligned} 0 &\neq [\frac{\partial F}{\partial u}(p) \times \frac{\partial F}{\partial v}(p)]_y \\ &= (\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial x}{\partial v})(p) \\ &= det\frac{\partial(x,z)}{\partial(u,v)}(p) \end{aligned}$$

Define $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ by $\pi(x,y,z) = (x,z)$, from the above we see that the map $\pi \circ F$ has nonzero Jacobian determinant at p. Thus by the Inverse Function Theorem $\pi \circ F$ has a smooth local inverse near p. In particular $(\pi \circ F)^{-1}$ is continuous near p. Therefore we conclude that $F^{-1} = (\pi \circ F)^{-1} \circ \pi$ is continuous near p. ■

Exercise 1.9.

a) Domain is $\mathbb{R}^2$. Range is $\mathbb{S}^2 \setminus \{(0,0,\pm 1)\}$

b) $\cos v$, $\sin v$, $\sinh u$, $\cosh u$ are smooth in the domain and F is smooth.

$$\begin{aligned} &\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} \\ &= \begin{bmatrix} i & j & k \\ -(\frac{\cos v}{\cosh^2 u})\sinh u & -(\frac{\sin v}{\cosh^2 u})\sinh u & \frac{1}{\cosh^2 u} \\ -\frac{\sin v}{\cosh u} & \frac{\cos v}{\cosh u} & 0 \end{bmatrix} \\ &= (-\frac{\cos v}{\cosh^3 u}, -\frac{\sin v}{\cosh^3 u}, -\frac{\sinh u}{\cosh^3 u}) \neq \mathbf{0} \qquad \forall u,v \in \mathbb{R} \end{aligned}$$

Let $F : U \to O$. Let $g(x,y,z) = x^2 + y^2 + z^2$, which is trivially smooth with $O \subset g^{-1}(1) \neq \phi$. $\bigtriangledown g = (2x, 2y, 2z)$ and $\bigtriangledown g = 0 \iff x = y = z = 0$ but $(0,0,0) \notin g^{-1}(1) \implies \bigtriangledown g \neq 0 \forall (x,y,z) \in g^{-1}(1)$. Hence by proposition 1.8, F is a homeomorphism and F is a smooth local parametrization.

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Exercise 1.10.

F is trivially smooth.

$$\begin{aligned} &\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} \\ &= \begin{bmatrix} i & j & k \\ -r\sin u\cos v & -r\sin u\sin v & r\cos u \\ -(r\cos u + R)\sin v & (r\cos u + R)\cos v & 0 \end{bmatrix} \\ &= (-r(r\cos u + R)\cos u\cos v, -r(r\cos u + R)\cos u\sin v, -r(r\cos u + R)\sin u). \end{aligned}$$

If $r(r\cos u + R)\sin u = 0$, then $r\cos u + R = 0$ or $\sin u = 0$, which implies $\cos u = -\frac{R}{r}$ or $u = \pi$.

- Case 1: $u = \pi$, $\cos u = -1$. Then $\frac{\partial F}{\partial v} = (r(-r+R)\cos v, r(-r+R)\sin v, 0)$. Since $\cos v, \sin v$ cannot equal 0 at the same time, $\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} \neq 0$.

- Case 2: $\cos u = -\frac{R}{r} < -1$ as $R > r > 0$. Hence not possible.

Therefore, $\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} \neq 0 \forall (u,v) \in (0,2\pi) \times (0,2\pi)$.

$$\begin{aligned}
&(R - \sqrt{[(r\cos u + R)\cos v]^2 + [(r\cos u + R)\sin v]^2})^2 + (r\sin u)^2 = r^2 \\
&\implies g(x,y,z) = (R - \sqrt{x^2+y^2})^2 + z^2
\end{aligned}$$

Let $F : U \to O$ and $\phi \neq O \subset g^{-1}(r^2)$. g is trivially smooth.

$$\begin{aligned}
&\bigtriangledown g = (2(R - \sqrt{x^2+y^2})(-\frac{x}{\sqrt{x^2+y^2}}), 2(R - \sqrt{x^2+y^2})(-\frac{y}{\sqrt{x^2+y^2}}), 2z) = 0 \\
&\iff z = 0 \\
&\iff (r - \sqrt{x^2+y^2})^2 = r^2 \\
&\iff \bigtriangledown g = (-2\frac{xr^2}{r^2+R}, -2\frac{yr^2}{r^2+R}, 0) \\
&\iff \begin{cases} x = 0 \\ y = 0 \end{cases}.
\end{aligned}$$

But x, y cannot equal 0 at the same time so $\bigtriangledown g \neq 0 \forall (x,y,z) \in g^{-1}(r^2)$. By proposition 1.8, F is a homeomorphism and hence F is a smooth local parametrization.

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Exercise 1.12.

Let $W = F((0,2\pi) \times \mathbb{R}) \cap \tilde{F}((-\pi,\pi) \times \mathbb{R})$. Then $\forall (\tilde{\theta}, \tilde{z}) \in \tilde{F}^{-1}(W)$,

$$\begin{aligned}
F^{-1}(\tilde{F}(\tilde{\theta}, \tilde{z})) &= F^{-1}(\cos\tilde{\theta}, \sin\tilde{\theta}, \tilde{z}) \\
&= \begin{cases} (\tilde{\theta}, \tilde{z}) \text{if } \tilde{\theta} \in (0,\pi) \\ (\tilde{\theta} + 2\pi, \tilde{z}) \text{if } \tilde{\theta} \in (-\pi, 0) \end{cases}
\end{aligned}$$

and $\forall (\theta, z) \in F^{-1}(W)$,

$$\begin{aligned}
\tilde{F}^{-1}(F(\theta, z)) &= \tilde{F}^{-1}(\cos\theta, \sin\theta, z) \\
&= \begin{cases} (\theta, z) \text{if } \theta \in (0,\pi) \\ (\theta - 2\pi, z) \text{if } \theta \in (\pi, 2\pi) \end{cases}
\end{aligned}$$

The max possible domains of $F^{-1} \circ \tilde{F}$ and $\tilde{F}^{-1} \circ F$ are $((-\pi,0) \cup (0,\pi)) \times \mathbb{R}$ and $((0,\pi) \cup \pi, 2\pi) \times \mathbb{R}$ respectively. In the result above, the transition maps are smooth. ■

Exercise 1.13.

The max possible domains of $F^{-1} \circ \tilde{F}$ and $\tilde{F}^{-1} \circ F$ are $(-1,1) \times ((-\pi,0) \cup (0,\pi))$ and $(-1,1) \times ((0,\pi) \cup (\pi, 2\pi))$ respectively.

$\forall (\tilde{u}, \tilde{\theta}) \in (-1,1) \times ((-\pi,0) \cup (0,\pi))$,

$$\begin{aligned}
F^{-1} \circ \tilde{F}(\tilde{u}, \tilde{\theta}) &= F^{-1}[(3 + \tilde{u}\cos\frac{\tilde{\theta}}{2})\cos\tilde{\theta}, (3 + \tilde{u}\cos\frac{\tilde{\theta}}{2})\sin\tilde{\theta}, \tilde{u}\sin\frac{\tilde{\theta}}{2}] \\
&= \begin{cases} (\tilde{u}, \tilde{\theta}) \text{if } \tilde{\theta} \in (0,\pi) \\ (-\tilde{u}, \tilde{\theta} + 2\pi) \text{if } \tilde{\theta} \in (-\pi, 0) \end{cases}
\end{aligned}$$

$\forall(u,\theta) \in (-1,1) \times ((0,\pi) \cup (\pi, 2\pi))$,

$$\begin{aligned}\tilde{F}^{-1} \circ F(u,\theta) &= \tilde{F}^{-1}[(3+u\cos\frac{\theta}{2})\cos\theta, (3+u\cos\frac{\theta}{2})\sin\theta, u\sin\frac{\theta}{2}] \\ &= \begin{cases} (u,\theta)\text{if } \theta \in (0,\pi) \\ (-u, \theta - 2\pi)\text{if } \theta \in (\pi, 2\pi) \end{cases}\end{aligned}$$

Hence the transition maps are smooth on their overlapping domain. ■

Exercise 1.14.

If $F_\beta^{-1} \circ F_\alpha$ is smooth, by similar argument, $F_\alpha^{-1} \circ F_\beta$ is also smooth. Also, as differentiability is a local property, for any $p \in W \subset M$, it suffices to show $F_\beta^{-1} \circ F_\alpha$ is smooth at $F_\alpha^{-1}(p)$.

$$\frac{\partial F_\alpha}{\partial u_1}(p) \times \frac{\partial F_\alpha}{\partial u_2}(p) \neq 0$$

and

$$\frac{\partial F_\alpha}{\partial u_1}(p) \times \frac{\partial F_\alpha}{\partial u_2}(p) = (det\frac{\partial(y,z)}{\partial(u_1,u_2)}(p), det\frac{\partial(z,x)}{\partial(u_1,u_2)}(p), det\frac{\partial(x,y)}{\partial(u_1,u_2)}(p))$$

Assume $det\frac{\partial(y,z)}{\partial(u_1,u_2)} \neq 0$. Since $\frac{\partial F_\alpha}{\partial u_1}(p) \times \frac{\partial F_\alpha}{\partial u_2}(p)$ and $\frac{\partial F_\beta}{\partial u_1}(p) \times \frac{\partial F_\beta}{\partial u_2}(p)$ are normal vectors to the surface at p, we deduce $det\frac{\partial(y,z)}{\partial v_1,v_2}(p) \neq 0$. Define $\pi(x,y,z) = (y,z)$, i.e. $\pi \circ F_\beta : U_\beta \to \mathbb{R}^2$ with $det\frac{\partial(y,z)}{\partial v_1,v_2}(p) \neq 0$. By inverse function thm, $(\pi \circ F_\beta)^{-1}$ exists and it is smooth near p. Then consider $F_\beta^{-1} \circ F_\alpha = (\pi \circ F_\beta^{-1}) \circ (\pi \circ F_\alpha)$ while $(\pi \circ F_\beta)^{-1}, \pi, F_\alpha$ are all smooth functions near p, the composition is also a smooth function near p. Since p is arbitrary, $F_\beta^{-1} \circ F_\alpha$ is smooth on $F_\alpha^{-1}(w)$. ■

Exercise 1.15.

Given M smooth surface. Given $F(u,v) = (u,v,\tilde{f}(u,v))$ is a smooth local parametrization. $f \circ F = \sqrt{(u-x_0)^2 + (v-y_0)^2 + (\tilde{f}(u,v) - z_0)^2}$ is smooth iff $(u,v,\tilde{f}(u,v)) \neq (x_0,y_0,z_0)$. Hence f is smooth iff $p_0 \notin M$. ■

Exercise 1.16.

$\mathbb{R}^2 - \{0\}$ is the domain of $F_+^{-1} \circ \Phi \circ F_+$ and $F_-^{-1} \circ \Phi \circ F_-$. $\mathbb{R}^2$ is the domain of $F_+^{-1} \circ \Phi \circ F_-$ and $F_-^{-1} \circ \Phi \circ F_+$.

- $\forall(u,v) \in \mathbb{R}^2 - \{0\}$,

$$\begin{aligned}F_+^{-1} \circ \Phi \circ F_+(u,v) &= F_+^{-1} \circ \Phi(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}) \\ &= F_+^{-1}(\frac{-2u}{u^2+v^2+1}, \frac{-2v}{u^2+v^2+1}, \frac{-u^2-v^2+1}{u^2+v^2+1}) \\ &= (-u,-v)\end{aligned}$$

- $\forall(u,v) \in \mathbb{R}^2 - \{0\}$,

$$\begin{aligned}F_-^{-1} \circ \Phi \circ F_-(u,v) &= F_-^{-1} \circ \Phi(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{1-u^2-v^2}{u^2+v^2+1}) \\ &= F_-^{-1}(\frac{-2u}{u^2+v^2+1}, \frac{-2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}) \\ &= (-u,-v)\end{aligned}$$

- $\forall (u,v) \in \mathbb{R}^2$,

$$\begin{aligned} F_+^{-1} \circ \Phi \circ F_-(u,v) &= F_+^{-1} \circ \Phi(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{1-u^2-v^2}{u^2+v^2+1}) \\ &= F_+^{-1}(\frac{-2u}{u^2+v^2+1}, \frac{-2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}) \\ &= (u,v) \end{aligned}$$

- $\forall (u,v) \in \mathbb{R}^2$,

$$\begin{aligned} F_-^{-1} \circ \Phi \circ F_+(u,v) &= F_-^{-1} \circ \Phi(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}) \\ &= F_-^{-1}(\frac{-2u}{u^2+v^2+1}, \frac{-2v}{u^2+v^2+1}, \frac{-u^2-v^2+1}{u^2+v^2+1}) \\ &= (u,v) \end{aligned}$$

Hence, all of them are smooth on their domains. ■

Exercise 1.20.

Let $F(x,y) = (x,y,f(x,y))$ be a smooth local parametrization around $p = (x_0, y_0, f(x_0,y_0))$. Then $\frac{\partial F}{\partial x}(p) = (1, 0, \frac{\partial f}{\partial x}|_{(x_0,y_0)})$ and $\frac{\partial F}{\partial y}(p) = (0, 1, \frac{\partial f}{\partial y}|_{(x_0,y_0)})$.
As the cross product of the two tangent vectors is perpendicular to the tangent plane, we simply take the cross product of the tangent vector to be one of a normal vector of the tangent plane. Hence,

$$\frac{\partial F}{\partial x}(p) \times \frac{\partial F}{\partial y}(p) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x}|_{(x_0,y_0)} \\ 0 & 1 & \frac{\partial f}{\partial y}|_{(x_0,y_0)} \end{vmatrix} = (-\frac{\partial f}{\partial x}|_{(x_0,y_0)}, -\frac{\partial f}{\partial y}|_{(x_0,y_0)}, 1)$$

Thus, $\forall (x,y,z) \in T_pM$,

$$\begin{aligned} &(\frac{\partial F}{\partial x}(p) \times \frac{\partial F}{\partial y}(p)) \cdot (x - x_0, y - y_0, z - f(x_0,y_0)) = 0 \\ &\Rightarrow (-\frac{\partial f}{\partial x}|_{(x_0,y_0)}, -\frac{\partial f}{\partial y}|_{(x_0,y_0)}, 1) \cdot (x - x_0, y - y_0, z - f(x_0,y_0)) = 0 \\ &\Rightarrow -\frac{\partial f}{\partial x}|_{(x_0,y_0)}(x - x_0) - \frac{\partial f}{\partial y}|_{(x_0,y_0)}(y - y_0) + (z - f(x_0,y_0)) = 0 \end{aligned}$$

Hence the equation of tangent plane is given by

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}|_{(x_0,y_0)}(y - y_0)$$

■

Exercise 1.21.

For $F(x,y) = (x, y, xf(\frac{y}{x}))$, we have

$$\begin{aligned} \frac{\partial F}{\partial x} &= (1, 0, f(\frac{y}{x}) - \frac{y}{x}f'(\frac{y}{x})) \\ \frac{\partial F}{\partial y} &= (0, 1, f'(\frac{y}{x})) \end{aligned}$$

Then for any $p = (a, b, af(\frac{b}{a}))$, we have

$$(a, b, af(\frac{b}{a})) = a\frac{\partial F}{\partial x}(p) + b\frac{\partial F}{\partial y}(p) \in p + T_pM$$

From the above we see that the position vector of p lies inside its tangent plane T_pM, which precisely means that the tangent plane passes through the origin. So this implies the conclusion. ■

Exercise 1.22.

(a) $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$ by $\Phi(x, y, z) = (xcos\ \alpha - ysin\ \alpha, xsin\ \alpha + ycos\ \alpha, z)$.

$$\begin{aligned} F(\theta, \varphi) &= (sin\ \varphi\ cos\ \theta, sin\ \varphi\ sin\ \theta, cos\ \varphi) \\ \Phi \circ F(\theta, \varphi) &= (sin\ \varphi\ cos\ \theta\ cos\ \alpha - sin\ \varphi\ sin\ \theta\ sin\ \alpha, \\ &\quad\ sin\ \varphi\ cos\ \theta\ sin\ \alpha + sin\ \varphi\ sin\ \theta\ cos\ \alpha, cos\ \varphi) \\ &= (sin\ \varphi\ (cos\ \theta\ cos\ \alpha - sin\ \theta\ sin\ \alpha), \\ &\quad\ sin\ \varphi\ (cos\ \theta\ sin\ \alpha + sin\ \theta\ cos\ \alpha), cos\ \varphi) \\ &= (sin\ \varphi\ cos\ (\theta + \alpha), sin\ \varphi\ sin\ (\theta + \alpha), cos\ \varphi) \end{aligned}$$

$$\begin{aligned} &\frac{\partial \Phi}{\partial \theta}(\Phi(p)) = \frac{\partial(\Phi \circ F)}{\partial \theta}(\theta, \varphi) = \frac{d}{dt}|_{t=0}(\Phi \circ F)(\theta + t, \varphi) \\ &= \frac{d}{dt}(sin\ \varphi\ cos\ (\theta + t + \alpha), sin\ \varphi\ sin\ (\theta + t + \alpha), cos\ \varphi)|_{t=0} \\ &= (-sin\ \varphi\ sin(\theta + t + \alpha), sin\ \varphi\ cos(\theta + t + \alpha), 0)|_{t=0} \\ &= (-sin\ \varphi\ sin(\theta + \alpha), sin\ \varphi\ cos(\theta + \alpha), 0) \end{aligned}$$

$$\begin{aligned} &\frac{\partial \Phi}{\partial \varphi}(\Phi(p)) = \frac{\partial(\Phi \circ F)}{\partial \varphi}(\theta, \varphi) = \frac{d}{dt}|_{t=0}(\Phi \circ F)(\theta, \varphi + t) \\ &= \frac{d}{dt}(sin\ (\varphi + t)\ cos\ (\theta + \alpha), sin\ (\varphi + t)\ sin\ (\theta + \alpha), cos\ (\varphi + t))|_{t=0} \\ &= (cos\ (\varphi + t)\ cos(\theta + \alpha), cos\ (\varphi + t)\ sin(\theta + \alpha), -sin(\varphi + t))|_{t=0} \\ &= (cos\ \varphi\ cos(\theta + \alpha), cos\ \varphi\ sin(\theta + \alpha), -sin\ \varphi) \end{aligned}$$

where (θ, φ) is a point in $(0, 2\pi)$x$(0, \pi)$ such that $F(\theta, \varphi) = p$.

(b) $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$ by $\Phi(x, y, z) = (xcos\ \alpha - ysin\ \alpha, xsin\ \alpha + ycos\ \alpha, z)$.

$$\begin{aligned} F_2(u, v) &= (u, v, -\sqrt{1 - u^2 - v^2}) \\ \Phi \circ F_2(u, v) &= (u\ cos\ \alpha - v\ sin\ \alpha, u\ sin\ \alpha + v\ cos\ \alpha, -\sqrt{1 - u^2 - v^2}) \end{aligned}$$

$$\begin{aligned} &\frac{\partial \Phi}{\partial u}(\Phi(p)) = \frac{\partial(\Phi \circ F_2)}{\partial u}(u, v) = \frac{d}{dt}|_{t=0}(\Phi \circ F_2)(u + t, v) \\ &= \frac{d}{dt}((u + t)\ cos\ \alpha - v\ sin\ \alpha, (u + t)\ sin\ \alpha + v\ cos\ \alpha, -\sqrt{1 - (u + t)^2 - v^2})|_{t=0} \\ &= (cos\ \alpha, sin\ \alpha, \frac{u + t}{\sqrt{1 - (u + t)^2 - v^2}})|_{t=0} \\ &= (cos\ \alpha, sin\ \alpha, \frac{u}{\sqrt{1 - u^2 - v^2}}) \end{aligned}$$

$$\frac{\partial \Phi}{\partial v}(\Phi(p)) = \frac{\partial(\Phi \circ F_2)}{\partial v}(u,v) = \frac{d}{dt}|_{t=0}(\Phi \circ F_2)(u, v+t)$$
$$= \frac{d}{dt}(u\ cos\ \alpha - (v+t)\ sin\ \alpha, u\ sin\ \alpha + (v+t)\ cos\ \alpha, -\sqrt{1-u^2-(v+t)^2})|_{t=0}$$
$$= (-sin\ \alpha, cos\ \alpha, \frac{v}{\sqrt{1-u^2-(v+t)^2}})|_{t=0}$$
$$= (-sin\ \alpha, cos\ \alpha, \frac{v}{\sqrt{1-u^2-v^2}})$$

where (u,v) is a point in $B_1(0)$ such that $F_2(u,v) = p$.

(c) $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$ by $\Phi(x,y,z) = (xcos\ \alpha - ysin\ \alpha, xsin\ \alpha + ycos\ \alpha, z)$

$$F_+(u,v) = (\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1})$$
$$= (\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, 1 - \frac{2}{u^2+v^2+1})$$
$$\Phi \circ F_+(u,v) = (\frac{2u}{u^2+v^2+1}\ cos\ \alpha - \frac{2v}{u^2+v^2+1}\ sin\ \alpha,$$
$$\frac{2u}{u^2+v^2+1}\ sin\ \alpha + \frac{2v}{u^2+v^2+1}\ cos\ \alpha, 1 - \frac{2}{u^2+v^2+1})$$

$$\frac{\partial \Phi}{\partial u}(\Phi(p)) = \frac{\partial(\Phi \circ F_+)}{\partial u}(u,v) = \frac{d}{dt}|_{t=0}(\Phi \circ F_+)(u+t, v)$$
$$= \frac{d}{dt}(\frac{2(u+t)}{(u+t)^2+v^2+1}\ cos\ \alpha - \frac{2v}{(u+t)^2+v^2+1}\ sin\ \alpha,$$
$$\frac{2(u+t)}{(u+t)^2+v^2+1}\ sin\ \alpha + \frac{2v}{(u+t)^2+v^2+1}\ cos\ \alpha, 1 - \frac{2}{(u+t)^2+v^2+1})|_{t=0}$$
$$= (\frac{-2(u+t)^2+2v^2+2}{((u+t)^2+v^2+1)^2}\ cos\ \alpha + \frac{4v(u+t)}{(u+t)^2+v^2+1}\ sin\ \alpha,$$
$$\frac{-2(u+t)^2+2v^2+2}{((u+t)^2+v^2+1)^2}\ sin\ \alpha - \frac{4v(u+t)}{(u+t)^2+v^2+1}\ cos\ \alpha, \frac{4(u+t)}{((u+t)^2+v^2+1)^2})|_{t=0}$$
$$= (\frac{-2u^2+2v^2+2}{(u^2+v^2+1)^2}\ cos\ \alpha + \frac{4vu}{u^2+v^2+1}\ sin\ \alpha,$$
$$\frac{-2u^2+2v^2+2}{(u^2+v^2+1)^2}\ sin\ \alpha - \frac{4vu}{u^2+v^2+1}\ cos\ \alpha, \frac{4u}{(u^2+v^2+1)^2})$$

$$
\begin{aligned}
&\frac{\partial \Phi}{\partial v}(\Phi(p)) = \frac{\partial(\Phi \circ F_+)}{\partial v}(u,v) = \frac{d}{dt}|_{t=0}(\Phi \circ F_+)(u, v+t) \\
&= \frac{d}{dt}(\frac{2u}{u^2+(v+t)^2+1}\ cos\ \alpha - \frac{2(v+t)}{u^2+(v+t)^2+1}\ sin\ \alpha, \\
&\quad \frac{2u}{u^2+(v+t)^2+1}\ sin\ \alpha + \frac{2(v+t)}{u^2+(v+t)^2+1}\ cos\ \alpha, 1 - \frac{2}{u^2+(v+t)^2+1})|_{t=0} \\
&= (\frac{-4u(v+t)}{u^2+(v+t)^2+1}\ cos\ \alpha - \frac{2u^2-2(v+t)^2+2}{(u^2+(v+t)^2+1)^2}\ sin\ \alpha, \\
&\quad -\frac{4u(v+t)}{u^2+(v+t)^2+1}\ sin\ \alpha + \frac{2u^2-2(v+t)^2+2}{(u^2+(v+t)^2+1)^2}\ cos\ \alpha, \frac{4(v+t)}{(u^2+(v+t)^2+1)^2})|_{t=0} \\
&= (\frac{-4uv}{u^2+v^2+1}\ cos\ \alpha - \frac{2u^2-2v^2+2}{(u^2+v^2+1)^2}\ sin\ \alpha, \\
&\quad -\frac{4uv}{u^2+v^2+1}\ sin\ \alpha + \frac{2u^2-2v^2+2}{(u^2+v^2+1)^2}\ cos\ \alpha, \frac{4v}{(u^2+v^2+1)^2})
\end{aligned}
$$

where (u, v) is a point in $\mathbb{R}^2$ such that $F_+(u, v) = p$.

■

Exercise 1.24.

From the expressions given in Example 1.5, we have

$$
\begin{aligned}
(v_1, v_2) &= F^{-1} \circ \Phi \circ F(u_1, u_2) \\
&= F^{-1} \circ \Phi(\frac{2u_1}{u_1^2+u_2^2+1}, \frac{2u_2}{u_1^2+u_2^2+1}, 1 - \frac{2}{u_1^2+u_2^2+1}) \\
&= F^{-1}(\frac{2u_1 cos\alpha - 2u_2 sin\alpha}{u_1^2+u_2^2+1}, \frac{2u_1 sin\alpha + 2u_2 cos\alpha}{u_1^2+u_2^2+1}, 1 - \frac{2}{u_1^2+u_2^2+1}) \\
&= (u_1 cos\alpha - u_2 sin\alpha, u_1 sin\alpha + u_2 cos\alpha)
\end{aligned}
$$

Therefore,

$$
(\Phi_*)_p = \frac{\partial(v_1, v_2)}{\partial(u_1, u_2)}\ |_{F^{-1}(p)} = \begin{bmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{bmatrix}
$$

■

Chapter 2

Exercise 2.5.

We have

$$F_3(v_1, \cdots, v_5) = [v_1 : v_2 : v_3 : 1 : v_4 : v_5],$$

$$F_1(u_1, \cdots, u_5) = [u_1 : 1 : u_2 : \cdots : u_5] = \left[\frac{u_1}{u_3} : \frac{1}{u_3} : \frac{u_2}{u_3} : 1 : \frac{u_4}{u_3} : \frac{u_5}{u_3}\right]$$

so $$F_3^{-1} \circ F_1(u_1, \cdots, u_5) = \left(\frac{u_1}{u_3}, \frac{1}{u_3}, \frac{u_2}{u_3}, \frac{u_4}{u_3}, \frac{u_5}{u_3}\right)$$

On the overlapping of $\mathcal{O}_1$ and $\mathcal{O}_3$, $u_3 \neq 0$, so in the domain of $F_3^{-1} \circ F_1$, $u_3 \neq 0$ and following this condition, it is clear that $F_3^{-1} \circ F_1$ is smooth. ■

Exercise 2.6.

$\mathbb{CP}^n = \{[x_0 + iy_0 : \cdots : x_n + iy_n] | (x_j, y_j) \in \mathbb{R}^2, \text{ not all } (x_j, y_j) = (0,0)\}$. Let

$$F_0(u_1, \cdots, u_n, v_1, \cdots, v_n) = [1 : u_1 + iv_1 : \cdots : u_n + iv_n]$$
$$F_1(u'_1, \cdots, u'_n, v'_1, \cdots, v'_n) = [u'_1 + iv'_1 : 1 : u'_2 + iv'_2 : \cdots : u'_n + iv'_n]$$

Note that $\{F_j | 0 \leq j \leq 1\}$ cover $\mathbb{CP}^n$ and, since $u'_1 + iv'_1 \neq 0$ in $F_1^{-1}(\mathcal{O}_0 \cap \mathcal{O}_1)$, the transition map

$$F_0^{-1} \circ F_1(u'_1, \cdots, u'_n, v'_1, \cdots, v'_n) = F^{-1}\left(\left[1 : \frac{1}{u'_1 + iv'_1} : \frac{u'_2 + iv'_2}{u'_1 + iv'_1} : \cdots : \frac{u'_n + iv'_n}{u'_1 + iv'_1}\right]\right)$$
$$= \left(\frac{u'_1}{u'^2_1 + v'^2_1}, \frac{u'_1 u'_2 + v'_1 v'_2}{u'^2_1 + v'^2_1}, \cdots, \frac{u'_1 u'_n + v'_1 v'_n}{u'^2_1 + v'^2_1}, \frac{-u'_1}{u'^2_1 + v'^2_1}, \frac{u'_1 v'_2 - v'_1 u'_2}{u'^2_1 + v'^2_1}, \cdots, \frac{u'_1 v'_n - v'_1 u'_n}{u'^2_1 + v'^2_1}\right)$$

is smooth as $u'_1 + uv'_1 \neq 0$. Other transition maps in the set of parametrizations can be generalized similarly. This concluded $\mathbb{CP}^n$ as a $2n$-manifold. ■

Exercise 2.9.

To prove that a smooth manifold has uncountably distinct differential structures, we need to show that it has uncountably incompatible atlas.

So in the first step, given a local paramatrization of the manifold, our goal is to construct an uncountable set of mutually incompatible local parametrization.

Then in the second step, for each local parametrization constructed in step 1, we want to construct an atlas (which is required to cover the whole manifold) generated by it. And by such a construction our goal would be achieved.

Let M be a smooth manifold, and let $F : U \to M$ be a local parametrization of it.
Up to a translation, we can assume $0 \in U$, by the openness of U we can find an open ball $B_\epsilon(0) \in U$.
Up to a reparametrization we can also let $\epsilon = 1$. We consider the map $\Psi_s : B_1(0) \to B_1(0)$ defined by $\Psi_s(u) = |u|^s u$ for $s > 0$. Then one can easily check that $\Psi_s(B_1(0)) = B_1(0)$. We note that Ψ_s is also a homeomorphism between its domain and image, hence the map $F_s = F \circ \Psi_s$ is also a local parametrization of M.

Let $V \subset B_1(0)$ be an open subset, $s_1 > s_2 > 0$, then we have

$$\begin{aligned}
&\Psi_{s_2}(B_1(0)) \subset \Psi_{s_1}(B_1(0)) \\
&\implies \Psi_{s_2}(V) \subset \Psi_{s_1}(B_1(0)) \\
&\implies F_{s_2}(V) \subset F_{s_1}(B_1(0)) \\
&\implies F_{s_2}(V) \subset F_{s_1}(B_1(0)) \cap F_{s_2}(B_1(0))
\end{aligned}$$

Hence the intersection $F_{s_1}(B_1(0)) \cap F_{s_2}(B_1(0))$ is nonempty and contains an open subset.

Now consider the transition map $F_{s_1}^{-1} \circ F_{s_2} : B_1(0) \to B_1(0)$, we let

$$\begin{aligned}
w &= F_{s_1}^{-1} \circ F_{s_2}(u) = \Psi_{s_1}^{-1}(|u|^{s_2} u) \\
&\implies |w|^{s_1} w = \Psi_{s_1}(w) = |u|^{s_2} u
\end{aligned}$$

from this we observe that $w = cu$ for some $c > 0$, solving for c we get $c = |u|^{\frac{s_2 - s_1}{1+s_1}}$. Thus,

$$\begin{aligned}
&F_{s_1}^{-1} \circ F_{s_2}(u) = |u|^{\frac{s_2-s_1}{1+s_1}} u \\
\implies &\frac{|F_{s_1}^{-1} \circ F_{s_2}(u) - F_{s_1}^{-1} \circ F_{s_2}(0)|}{|u|} = |u|^{\frac{s_2-s_1}{1+s_1}}
\end{aligned}$$

Since $\lim_{u \to 0} |u|^{\frac{s_2-s_1}{1+s_1}}$ diverges, the transition map $F_{s_1}^{-1} \circ F_{s_2}$ is not smooth, this implies that $F \circ \Psi_{s_1}$ and $F \circ \Psi_{s_2}$ are not compatible whenever $s_1 \neq s_2$.
Moreover, suppose that $A = \{F_i\}$ is an atlas of M, then $M = \bigcup_i F_i(U_i)$. We let $U_i = \bigcup_j B_{ij}$ be a countable union of open balls, and define the maps $\Phi_{ij} : B_1(0) \to B_{ij}$ be homeomophisms between $B_1(0)$ and B_{ij}. Let $F_{ij} = F_i \circ \Phi_{ij}$, then we observe that

$$F_{ij}(B_1(0)) = F_i(B_{ij}) \implies \bigcup_{ij} F_{ij}(B_1(0)) = \bigcup_{ij} F_i(B_{ij}) = \bigcup_i F_i(\cup_j B_{ij}) = M.$$

By the fact that $\Psi_s(B_1(0)) = B_1(0)$, we have $\bigcup_{ij} F_{ij} \circ \Psi_s(B_1(0)) = M$ for any $s > 0$.
Hence $A_s = \{F_{ij} \circ \Psi_s\}$ is also an atlas of M for any $s > 0$. But from the discussion above we see that the two altas A_{s_1} and A_{s_2} have distinct differential structures whenever $s_1 \neq s_2$.
Therefore we conclude that M has uncountably distinct differential structures. ■

Exercise 2.13.

The square $(-1,1) \times (-1,1)$ can be parametrized by the identity map $\mathrm{Id}_\square(u,v) = (u,v)$ and similarly $\mathbb{R}^2$ can be parametrized by the identity map $\mathrm{Id}_{\mathbb{R}^2}(x,y) = (x,y)$. Consider the map $\Phi : (-1,1) \times (-1,1) \to \mathbb{R}^2$ defined by $\Phi(x_1, x_2) = (\tan(\pi x_1/2), \tan(\pi x_2/2))$. The map maps $(-1,1) \times (-1,1)$ to $\mathbb{R}^2$ and the transition map $\mathrm{Id}_{\mathbb{R}^2}^{-1} \circ \Phi \circ \mathrm{Id}_\square(u,v) = (\tan(\pi u/2), \tan(\pi v/2))$ is smooth on the domain. Meanwhile, $\det D(\mathrm{Id}_{\mathbb{R}^2}^{-1} \circ \Phi \circ \mathrm{Id}_\square(u,v)) = \frac{\pi^2}{4} \sec^2(\pi u/2) \sec^2(\pi v/2) \neq 0$ on $(-1,1) \times (-1,1)$. Therefore $\mathrm{Id}_{\mathbb{R}^2}^{-1} \circ \Phi \circ \mathrm{Id}_\square$ is invertible and smooth by the inverse function theorem. This proved Φ as the diffeomorphism between $(-1,1) \times (-1,1)$ and $\mathbb{R}^2$. So they are diffeomorphic. ■

Exercise 2.16.

$$\frac{\partial}{\partial x_2} = \frac{\partial y_0}{\partial x_2} \frac{\partial}{\partial y_0} + \frac{\partial y_2}{\partial x_2} \frac{\partial}{\partial y_2} = \frac{1}{x_1} \frac{\partial}{\partial y_2} = y_0 \frac{\partial}{\partial y_2}$$

■

Exercise 2.17.

With the defined F_1, F_2, we have

$$(y_1, y_2) = F_2^{-1} \circ F_1(x_1, x_2) = \left(\frac{x_1}{x_1^2 + x_2^2}, -\frac{x_2}{x_1^2 + x_2^2}\right)$$

from chain rule, we have

$$\partial_{x_1} = \frac{\partial y_1}{\partial x_1}\partial_{y_1} + \frac{\partial y_2}{\partial x_1}\partial_{y_2} = \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2}\partial_{y_1} + \frac{2x_2x_1}{(x_1^2 + x_2^2)^2}\partial_{y_2} = (y_2^2 - y_1^2)\partial_{y_1} - 2y_1y_2\partial_{y_2}$$
$$\partial_{x_2} = \frac{\partial y_1}{\partial x_2}\partial_{y_1} + \frac{\partial y_2}{\partial x_2}\partial_{y_2} = -\frac{2x_2x_1}{(x_1^2 + x_2^2)^2}\partial_{y_1} + \frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2}\partial_{y_2} = 2y_1y_2\partial_{y_1} + (y_1^2 - y_2^2)\partial_{y_2}$$

■

Exercise 2.19.

$$\begin{aligned}
&H^{-1} \circ \Phi(F \times G)(u, v_0, v_1) \\
&= H^{-1} \circ \Phi([u : 1], [v_0 : 1 : v_2]) \\
&= H^{-1} \circ ([uv_0 : u : uv_2 : v_0 : 1 : v_2]) \\
&= \left(\frac{v_0}{v_2}, \frac{1}{v_2}, \frac{v_0}{uv_2}, \frac{1}{uv_2}, \frac{1}{u}\right) \\
\Rightarrow [\Phi_*] &= \begin{bmatrix} 0 & \frac{1}{v_2} & -\frac{v_0}{v_2^2} \\ 0 & 0 & -\frac{1}{v_2^2} \\ -\frac{v_0}{u^2v_2} & \frac{1}{uv_2} & -\frac{v_0}{uv_2^2} \\ -\frac{1}{u^2v_2} & 0 & -\frac{1}{uv_2^2} \\ -\frac{1}{u^2} & 0 & 0 \end{bmatrix}
\end{aligned}$$

■

Exercise 2.21.

Without loss of generality, we can assume M is an n-manifold with set of local parameterization $\{F_i\}$. Since $F_i^{-1} \circ Id_M \circ F_i(u_1, \cdots, u_n) = (u_1, \cdots, u_n)$ so for any $p \in M$ there exist F_i as a parameterization. Through sandwiching Id_M in between the same parameterization, $[\Phi_*] = I_{n\times n}$, and this true for every $p \in M$. ■

Exercise 2.24.

Let $I : \mathbb{R}^4 \to \mathbb{R}^4$ be the identity map. We have

$$I^{-1} \circ \Phi \circ G(y_1, z_1) = \left(1 - y_1^2, y_1, z_1, y_1z_1\right) = \left(1 - y_1^2, y_1, z_1, y_1z_1\right)$$
$$\Rightarrow [\Phi_*] = \begin{bmatrix} -2y_1 & 0 \\ 1 & 0 \\ 0 & 1 \\ z_1 & y_1 \end{bmatrix} = \begin{bmatrix} -\frac{2v}{u} & 0 \\ 1 & 0 \\ 0 & 1 \\ \frac{u^2+v^2-1}{2u} & \frac{v}{u} \end{bmatrix}$$

[Using $y_1 = y/x$, $z_1 = z/x$ of (x, y, z) from stereographic projection]. Moreover,

$$\begin{aligned} G^{-1} \circ \pi \circ F(u, v) &= \left(\frac{v}{u}, \frac{u^2 + v^2 - 1}{2u} \right) \\ \Rightarrow [\pi_*] &= \begin{bmatrix} -\frac{v}{u^2} & \frac{1}{u} \\ \frac{u^2 - v^2 + 1}{2u^2} & \frac{v}{u} \end{bmatrix} \\ \Rightarrow [\Phi_*][\pi_*] &= \begin{bmatrix} -\frac{2v^2}{u^3} & -\frac{2v}{u^2} \\ -\frac{v}{u^2} & \frac{1}{u} \\ \frac{u^2 - v^2 + 1}{2u^2} & \frac{v}{u} \\ -\frac{v^3 + v}{u^3} & \frac{u^2 + 3v^2 - 1}{2u^2} \end{bmatrix} \end{aligned}$$

Meanwhile,

$$\begin{aligned} I^{-1} \circ (\Phi \circ \pi) \circ F(u, v) &= \left(1 - \frac{v^2}{u^2}, \frac{v}{u}, \frac{u^2 + v^2 - 1}{2u}, \frac{vu^2 + v^3 - v}{2u^2} \right) \\ \Rightarrow [(\Phi \circ \pi)_*] &= \begin{bmatrix} -\frac{2v^2}{u^3} & -\frac{2v}{u^2} \\ -\frac{v}{u^2} & \frac{1}{u} \\ \frac{u^2 - v^2 + 1}{2u^2} & \frac{v}{u} \\ -\frac{v^3 + v}{u^3} & \frac{u^2 + 3v^2 - 1}{2u^2} \end{bmatrix} \end{aligned}$$

This verified $[(\Phi \circ \pi)_*] = [\Phi_*][\pi_*]$. ■

Exercise 2.27.

Note that $[\Phi_*] = \begin{bmatrix} 3x^2 & 0 \\ 2xy & x^2 \\ y^2 & 2xy \\ 0 & 3y^2 \end{bmatrix}$.

Given $(x, y) \neq (0, 0)$, in case $x \neq 0$, the first column have a pivot at first row and second column have a pivot at the second row. So $[\Phi_*]$ is injective.
In case $y \neq 0$, $[\Phi_*]$ has a pivot at the fourth row of the second column and a pivot at the third row of the first column.
In either case, $[\Phi_*]$ is injective, so we conclude Φ is an immersion at $(x, y) \neq (0, 0)$.

■

Exercise 2.28.

From chain rule, $(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*$. Ψ_*, Φ_* are injective by the definition of immersion. Since composition of injective map is injective, we conclude $(\Psi \circ \Phi)_*$ is injective. So the map $\Psi \circ \Phi$ is also an immersion.

■

Exercise 2.31.

(i)⇒(ii),(iii) Given Φ is a local diffeomorphism, on the concerned domain of Φ, $[\Phi_*]$ is an invertible matrix and therefore it is automatically surjective and injective. This implies Φ is a submersion and is an immersion.

(ii)⇒(i) Given Φ is an immersion, on the concerned domain of Φ, $[\Phi_*]$ every column of $[\Phi_*]$ is a pivot. Since $\dim M = \dim N$, number of columns = number of rows. So every column and every row is a pivot. This implies Φ is a local diffeomorphism.

(iii)⇒(i) Given Φ is a submersion, on the concerned domain of Φ, $[\Phi_*]$ every column of $[\Phi_*]$ is a pivot. Since $\dim M = \dim N$, number of rows = number of columns. So every row and every column is a pivot. This implies Φ is a local diffeomorphism. ■

Exercise 2.33.

Let $F_0(x_1, \cdots, x_n) = [1 : x_1 : \cdots : x_n]$, $Id : (u_0, \cdots, u_n) \in \mathbb{R}^{n+1} \mapsto (u_0, \cdots, u_n) \in \mathbb{R}^{n+1}$. Then

$$F_0^{-1} \circ \Phi \circ Id(x_0, \cdots, x_n) = \left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right)$$

since $x_0 = 0$ is not in the domain of this transition map, for this parameterization, let $\vec{e} = [x_1 \quad \cdots \quad x_n]^T$. Then

$$[\Phi_*] = \begin{bmatrix} -\frac{1}{x_0^2}\vec{e} & \frac{1}{x_0}I_{n\times n} \end{bmatrix}$$

so every row is a pivot. Other parameterization can be generalized similarly and this implies $[\Phi_*]$ is surjective. Therefore Φ is a submersion. ■

Exercise 2.38.

Let $F(u) = (u, |u|)$ be a parametrization of Γ and $G(v, w) = (v, w+|v|)$ be a parametrization of $\mathbb{R}^2$, then the transition map of the inclusion map is given by: $G^{-1} \circ \iota \circ F(u) = G^{-1}(u, |u|) = (u, 0)$ which is clearly smooth and meanwhile, $[\iota_*] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which is injective. So ι is a smooth immersion. Therefore, together we can "conclude" Γ as a submanifold of $\mathbb{R}^2$. ■

Chapter 3

Exercise 3.1.

(a) $\forall v \in V, \forall \alpha \in \mathbb{R}, \forall f, g \in V^*$

1. (Closed in addition) $\forall f, g \in V^*, \forall \alpha \in \mathbb{R} \forall u, v \in V$,

$$\begin{aligned} (f+g)(\alpha u + v) &= f(\alpha u + v) + g(\alpha u + v) \\ &= \alpha f(u) + f(v) + \alpha g(u) + g(v) \\ &= \alpha (f+g)(u) + (f+g)(v) \\ \Rightarrow f + g \in V^* \end{aligned}$$

2. (Commutativity) $\forall f, g \in V^*, \forall v \in V$,

$$\begin{gathered} (f+g)(v) = f(v) + g(v) = g(v) + f(v) = (g+f)(v) \\ \Rightarrow f + g = g + v \end{gathered}$$

3. (Associativity in addition) $\forall f, g \in V^*, \forall v \in V$,

$$\begin{aligned} ((f+g)+h)(v) &= (f+g)(v) + h(v) \\ &= (f(v) + g(v)) + h(v) \\ &= f(v) + (g(v) + h(v)) \\ &= f(v) + (g+h)(v) = (f + (g+h))(v) \\ \Rightarrow (f+g)+h &= f + (g+h) \end{aligned}$$

4. (Zero vector) $\forall f \in V^*, \forall v \in V, \exists 0(v) \in V^*$,

$$\begin{gathered} (f+0)(v) = f(v) + 0(v) = f(v) + 0 = f(v) \\ \Rightarrow f + 0 = f \end{gathered}$$

5. (Negative) $\forall f \in V^*, \forall v \in V, \exists - f \in V^*$

$$\begin{gathered} (f + (-f))(v) = f(v) + (-f)(v) = f(v) - f(v) = 0 = 0(v) \\ \Rightarrow f + (-f) = 0 \end{gathered}$$

6. (Closed in scalar multiplcation) $\forall f \in V^*, \forall \alpha, \beta \in \mathbb{R} \forall u, v \in V$,

$$\begin{gathered} (\alpha f)(\beta u + v) = (\alpha)(f(\beta u + v)) = \alpha(\beta f(u) + f(v)) = \beta(\alpha f)(u) + (\alpha f)(v) \\ \Rightarrow \alpha f \in V^* \end{gathered}$$

7. (Associativity in scalar multiplcation) $\forall f \in V^*, \forall \alpha, \beta \in \mathbb{R} \forall v \in V$,

$$((\alpha\beta)f)(v) = (\alpha(\beta f))(v)) \Rightarrow (\alpha\beta)f = \alpha(\beta f)$$

8. (Identity scalar) $\forall f \in V^*, \exists 1 \in \mathbb{R} \forall v \in V$, $(1 \times f)(v) = f(v) \Rightarrow (1)f = f$
9. (Distributivity of vectors) $\forall f, g \in V^*, \forall \alpha \in \mathbb{R} \forall v \in V$,

$$\begin{aligned} \alpha(f+g)(v) &= \alpha(f(v) + g(v)) = \alpha f(v) + \alpha g(v) = (\alpha f)(v) + (\alpha g)(v) \\ &= (\alpha f + \alpha g)(v) \Rightarrow \alpha(f+g) = \alpha f + \alpha g \end{aligned}$$

10. (Distributivity of scalars) $\forall f \in V^*, \forall \alpha, \beta \in \mathbb{R} \forall v \in V$,

$$(\alpha + \beta) f(v) = \alpha f(v) + \beta f(v) = (\alpha f + \beta f)(v)$$
$$\Rightarrow (\alpha + \beta) f = \alpha f + \beta f$$

$\therefore V^*$ is a vector space.

I will show (c) then show (b).

(c) $\{e_1^*, ..., e_n^*\}$ be the basis of V^*. We have $\sum_{i=1}^n a_i e_i^* = 0$. Sub

$$x = e_j \Rightarrow \sum_{i=1}^n a_i e_i^*(e_j) = 0(e_j) \Rightarrow \sum_{i=1}^n a_i \delta_{ij} = 0 \Rightarrow a_j = 0$$

for $j = 1, 2, ..., n$. Hence $\{e_i^*\}_{i=1}^n$ are linearly independent.
Obviously, $\text{Span}\{e_i^*\}_{i=1}^n \subseteq V^*$. As

$$\forall T \in V^*, \sum_{j=1}^n T(e_j) e_j^*(e_i) = \sum_{j=1}^n T(e_j) \delta_{ji} = T(e_i)$$
$$\forall T \in V^*, T = \sum_{i=1}^n T(e_i) e_i^* \in \text{Span}\{e_i^*\}_{i=1}^n$$

Thus $\text{Span}\{e_i^*\}_{i=1}^n = V^*$.
Thus$\{e_i^*\}_{i=1}^n$ is a basis for V^*.

(b) As from the result of part(c), it directly implies the number of basis in V is same as the number of basis in dual of V. Hence $dimV = dimV^*$.
Verification: $T \in V^*, T = \sum_{i=1}^n c_i e_i^*$. Sub $x = e_i$, we have

$$T(e_i) = \sum_{j=1}^n c_j e_j^*(e_i) = \sum_{j=1}^n c_j \delta_{ji} = c_i = a_i$$

Hence $T = \sum_{i=1}^n a_i e_i^*$.

■

Exercise 3.2.

Using the formula (3.1) in lecture note p.63 , we can express $\{dx, dy, dz\}$ in terms of $\{dr, d\theta, dz\}$ and $\{d\rho, d\phi, d\theta\}$. Ans:

$$\begin{aligned}\{dx, dy, dz\} &= \{\cos\theta dr - r\sin\theta d\theta, \sin\theta dr + r\cos\theta d\theta, dz\} \\ \{dx, dy, dz\} &= \{\sin\phi\cos\theta d\rho + \rho\cos\phi\cos\theta d\phi - \rho\sin\phi\sin\theta d\theta, \\ &\qquad \sin\phi\sin\theta d\rho + \rho\cos\phi\sin\theta d\phi + \rho\sin\phi\cos\theta d\theta, \\ &\qquad \cos\phi d\rho - \rho\sin\phi d\phi\}\end{aligned}$$

■

Exercise 3.3.

$$\omega = \sum_j a_j du^j = \sum_j a_j \Big(\sum_i \frac{\partial u_j}{\partial v_i} dv^i\Big) = \sum_i \Big(\sum_j a_j \frac{\partial u_j}{\partial v_i}\Big) dv^i = \sum_i b_i dv^i \Rightarrow b_i = \sum_j a_j \frac{\partial u_j}{\partial v_i}$$

■

Exercise 3.7.

Suppose p is a point at U_M, then by chain rule,

$$\Phi_*(\frac{\partial}{\partial u_i}) = \frac{\partial \Phi}{\partial u_i} = \sum_{j=1}^{n} \frac{\partial v_j}{\partial u_i}\frac{\partial}{\partial v_j}$$

We can then compute

$$\Phi^* dv^k(\frac{\partial}{\partial u_i}) = dv^k(\Phi_*(\frac{\partial}{\partial u_i})) = dv^k \sum_{j=1}^{n} \frac{\partial v_j}{\partial u_i}\frac{\partial}{\partial v_j}(\Phi(p)) = \frac{\partial v_k}{\partial u_i}$$

Summing up all the contribution of u_i's, we have $\Phi^* dv^k = \sum_{i=1}^{m} \frac{\partial v_k}{\partial u_i} du^i$. ■

Exercise 3.8.

Consider

$$\begin{aligned}
&(\Phi^* dy)(\partial_r) = dy(\Phi_*\partial_r) = dy(cos(\theta)\partial_x + sin(\theta)\partial_y)) = \sin\theta \\
&(\Phi^* dy)(\partial_\theta) = dy(\Phi_*\partial_\theta) = dy(-y\partial_x + x\partial_y) = x = r\cos\theta \\
&\Rightarrow \Phi^* dy = \sin\theta dr + r\cos\theta d\theta
\end{aligned}$$

Meanwhile, using (3.4)

$$\Phi^* dy = \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial r} dr = r\cos\theta d\theta + \sin\theta dr$$

which agrees with the first expression. ■

Exercise 3.9.

We may employ (3.4) to compute the local expression for Φ^*.

$$\begin{aligned}
\Phi^*(dy^1) &= \sum_{i=1}^{2} \frac{\partial y_1}{\partial x_i} dx_i = x_2 dx_1 + x_1 dx_2 \\
\Phi^*(dy^2) &= \sum_{i=1}^{2} \frac{\partial y_2}{\partial x_i} dx_i = x_3 dx_2 \\
\Phi^*(dy^3) &= \sum_{i=1}^{2} \frac{\partial y_3}{\partial x_i} dx_i = x_3 dx_1
\end{aligned}$$

■

Exercise 3.10.

First, we compute

$$F^{-1} \circ \Phi(x, y, z) = F^{-1}([x : y : z]) = F^{-1}([1 : y/x, z/x]) = (y/x, z/x) = (u_1, u_2)$$

Thus, we have

$$\begin{aligned}
\Phi^*(du^1) &= \sum_{i=1}^{3} \frac{\partial u_1}{\partial x_j} dx^j = \frac{-y}{x^2} dx + \frac{1}{x} dy \\
\Phi^*(du^2) &= \sum_{i=1}^{3} \frac{\partial u_2}{\partial x_j} dx^j = \frac{-z}{x^2} dx + \frac{1}{x} dz
\end{aligned}$$

where the $(x_1, x_2, x_3) \equiv (x, y, z)$. ■

Exercise 3.20.

We perform the computation using the definition:

$$\alpha T \otimes S(X, Y) = \alpha T(X)S(Y)$$

while

$$T \otimes (\alpha S)(X, Y) = T(X)(\alpha S)(Y) = T(X)\alpha S(Y) = \alpha T(X)S(Y)$$

so the two expressions are indeed the same. ■

Exercise 3.21.

We have $T \otimes (\alpha_1 S_1 + \alpha_2 S_2)(X, Y) = T(X)(\alpha_1 S_1 + \alpha_2 S_2)(Y)$. Since S is linear, we have

$$(\alpha_1 S_1 + \alpha_2 S_2)(Y) = \alpha_1 S_1(Y) + \alpha_2 S_2(Y)$$

Therefore

$$\begin{aligned}
&T \otimes (\alpha_1 S_1 + \alpha_2 S_2)(X, Y) \\
&= T(X)(\alpha_1 S_1(Y) + \alpha_2 S_2(Y)) \\
&= \alpha_1 T(X)S_1(Y) + \alpha_2 T(X)S_2(Y) \\
&= \alpha_1 T \otimes S_1(X, Y) + \alpha_2 T \otimes S_2(X, Y) \\
&= (\alpha_1 T \otimes S_1 + \alpha_2 T \otimes S_2)(X, Y)
\end{aligned}$$

Similar procedures apply for the T slot. ■

Exercise 3.22.

Assume $\sum_{i,j=1}^{n} a_{ij} e_i^* \otimes e_j^* = 0$. Then

$$0 = \sum_{i,j=1}^{n} a_{ij} e_i^* \otimes e_j^*(e_k, e_l) = \sum_{i,j=1}^{n} a_{ij} e_i^*(e_k) e_j^*(e_l) = \sum_{i,j=1}^{n} a_{ij}\delta_{ik}\delta_{jl} = a_{kl}$$

for all $k, l = 1, 2, ..., n$. Hence $\{e_i^* \otimes e_j^*\}_{i,j=1}^{n}$ are linear independent.

Let $\omega \in V^* \otimes V^*$, As

$$\sum_{i,j=1}^{n} \omega(e_i, e_j) e_i^* \otimes e_j^*(e_k, e_l) = \sum_{i,j=1}^{n} \omega(e_i, e_j) e_i^*(e_k) e_j^*(e_l) = \omega(e_k, e_l)$$

for all k, l, hence $\omega = \sum_{i,j=1}^{n} \omega(e_i, e_j) e_i^* \otimes e_j^*$ and Span $\{e_i^* \otimes e_j^*\}_{i,j=1}^{n}$=$V^* \otimes V^*$.

Thus $\{e_i^* \otimes e_j^*\}_{i,j=1}^{n}$ is a basis of $V^* \otimes V^*$. The dimesnion of $V^* \otimes V^* = n^2$. ■

Exercise 3.23.

Suppose ω is a linear combination of $e_i^* \otimes e_j^*$.
From $\omega(e_1, e_1) = \omega(e_2, e_2) = 0$, we can deduce that the only possibilities for i, j are $(i, j) = (2, 1)$ or $(i, j) = (1, 2)$. Using the remaining 2 conditions, we can deduce

$$\omega = 3e_1^* \otimes e_2^* - 3e_2^* \otimes e_1^*$$

■

Exercise 3.24.

(a) not well-defined since the domain of T_1 is $T_pM \times T_pM$.

(b) $T_2(\frac{\partial}{\partial u_1}) = du^1(\frac{\partial}{\partial u_1})\frac{\partial}{\partial u_2} = \frac{\partial}{\partial u_2}$

(c) $T_1(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}) = du_1(\frac{\partial}{\partial u_1})du_2(\frac{\partial}{\partial u_2}) = 1$

(d) not well-defined since the domain of T_2 is T_pM.

■

Exercise 3.25.

One can check that the required tensor product expressions of T are:

(i) The first line: $T = du_1 \otimes (\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2})$.

(ii) The second line: $T = du_2 \otimes (\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2})$

■

Exercise 3.26.

Using the expressions of (u, v) in (3.21), one may compute

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{-2xy}{(x^2+y^2)^2}\\ \frac{\partial v}{\partial y} &= \frac{y^2-x^2}{(x^2+y^2)^2}\end{aligned}$$

We then have

$$\begin{aligned}\frac{\partial}{\partial y} &= \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\\ &= \frac{-2xy}{(x^2+y^2)^2}\frac{\partial}{\partial u} + \frac{y^2-x^2}{(x^2+y^2)^2}\frac{\partial}{\partial v}\\ &= (2uv)\frac{\partial}{\partial u} + (v^2-u^2)\frac{\partial}{\partial v}\end{aligned}$$

Together with the expression of dx in the notes, we have

$$\begin{aligned}dx \otimes \frac{\partial}{\partial y} &= (\frac{-2uv}{(u^2+v^2)^2}du + \frac{v^2-u^2}{(u^2+v^2)^2}dv) \otimes ((2uv)\frac{\partial}{\partial u} + (v^2-u^2)\frac{\partial}{\partial v})\\ &= \frac{-(2uv)^2}{(u^2+v^2)^2}du \otimes \frac{\partial}{\partial u} + \frac{-2uv(v^2-u^2)}{(u^2+v^2)^2}du \otimes \frac{\partial}{\partial v}\\ &\quad + \frac{2uv(v^2-u^2)}{(u^2+v^2)^2}dv \otimes \frac{\partial}{\partial u} + \frac{(v^2-u^2)^2}{(u^2+v^2)^2}dv \otimes \frac{\partial}{\partial v}\end{aligned}$$

Meanwhile $x^2 + y^2 = \frac{1}{u^2+v^2}$. Thus, we have

$$\begin{aligned}\Omega = e^{\frac{-1}{u^2+v^2}}(&\frac{-(2uv)^2}{(u^2+v^2)^2}du \otimes \frac{\partial}{\partial u} + \frac{-2uv(v^2-u^2)}{(u^2+v^2)^2}du \otimes \frac{\partial}{\partial v}\\ &+ \frac{2uv(v^2-u^2)}{(u^2+v^2)^2}dv \otimes \frac{\partial}{\partial u} + \frac{(v^2-u^2)^2}{(u^2+v^2)^2}dv \otimes \frac{\partial}{\partial v})\end{aligned}$$

■

Exercise 3.27.

$$\begin{aligned}
Rm &= \sum_{\alpha,\beta,\gamma,\eta} \tilde{R}^{\eta}_{\alpha\beta\gamma} dv^{\alpha} \otimes dv^{\beta} \otimes dv^{\gamma} \otimes \frac{\partial}{\partial v^{\eta}} \\
&= \sum_{\alpha,\beta,\gamma,\eta} \tilde{R}^{\eta}_{\alpha\beta\gamma} (\sum_i \frac{\partial v^{\alpha}}{\partial u^i} du^i) \otimes (\sum_j \frac{\partial v^{\beta}}{\partial u^j} du^j) \otimes (\sum_k \frac{\partial v^{\gamma}}{\partial u^k} du^k) \otimes (\sum_l \frac{\partial u^l}{\partial v^{\eta}} \frac{\partial}{\partial u^l}) \\
&= \sum_{i,j,k,l} (\sum_{\alpha,\beta,\gamma,\eta} \frac{\partial v^{\alpha}}{\partial u^i} \frac{\partial v^{\beta}}{\partial u^j} \frac{\partial v^{\gamma}}{\partial u^k} \frac{\partial u^l}{\partial v^{\eta}} \tilde{R}^{\eta}_{\alpha\beta\gamma}) du^i \otimes du^j \otimes du^k \otimes \frac{\partial}{\partial u^l} \\
\therefore R^l_{ijk} &= \sum_{\alpha,\beta,\gamma,\eta} \frac{\partial v^{\alpha}}{\partial u^i} \frac{\partial v^{\beta}}{\partial u^j} \frac{\partial v^{\gamma}}{\partial u^k} \frac{\partial u^l}{\partial v^{\eta}} \tilde{R}^{\eta}_{\alpha\beta\gamma}
\end{aligned}$$

■

Exercise 3.32.

$$\begin{aligned}
&T1 \wedge T2 \wedge T3 \wedge T4 \\
&= T1 \otimes T2 \otimes T3 \otimes T4 - T1 \otimes T2 \otimes T4 \otimes T3 - T1 \otimes T3 \otimes T2 \otimes T4 \\
&+ T1 \otimes T3 \otimes T4 \otimes T2 + T1 \otimes T4 \otimes T2 \otimes T3 - T1 \otimes T4 \otimes T3 \otimes T2 \\
&- T2 \otimes T1 \otimes T3 \otimes T4 + T2 \otimes T1 \otimes T4 \otimes T3 + T2 \otimes T3 \otimes T1 \otimes T4 \\
&- T2 \otimes T3 \otimes T4 \otimes T1 + T2 \otimes T4 \otimes T1 \otimes T3 - T2 \otimes T4 \otimes T3 \otimes T1 \\
&+ T3 \otimes T1 \otimes T2 \otimes T4 - T3 \otimes T1 \otimes T4 \otimes T2 - T3 \otimes T2 \otimes T1 \otimes T4 \\
&+ T3 \otimes T2 \otimes T4 \otimes T1 + T3 \otimes T4 \otimes T1 \otimes T2 - T3 \otimes T4 \otimes T2 \otimes T1 \\
&- T4 \otimes T1 \otimes T2 \otimes T3 + T4 \otimes T1 \otimes T3 \otimes T2 + T4 \otimes T2 \otimes T1 \otimes T3 \\
&- T4 \otimes T2 \otimes T3 \otimes T1 - T4 \otimes T3 \otimes T1 \otimes T2 + T4 \otimes T3 \otimes T2 \otimes T1
\end{aligned}$$

■

Exercise 3.33.

By definition, the space $\wedge^k V^* = \text{span}\{e^*_{i_1} \wedge ... \wedge e^*_{i_k}\}$, where $e^*_{i_m}, m = 1, ...k$ are k linearly independent vectors in $V^* = \text{span}\{e^*_1, ..., e^*_n\}$ if $0 \le k \le n$. For $k = 0$, by definition $\wedge^0 V^* = \mathbb{R}$, which has dimension $1 = C^n_0$.

For $0 < k \le n$, consider the wedge product $e^*_{i_1} \wedge ... \wedge e^*_{i_k}$. If we permute the $i_1, ..., i_k$, we will have an extra ± 1 factor only, so the new vector in $\wedge^k V^*$ is linearly dependent to the original one. Therefore, only different combinations give a new vector that is linearly independent to the original one. Since we can choose k vectors in $\{e^*_1, ..., e^*_n\}$ to form the wedge product, we will have C^n_k linearly independent vectors in $\wedge^k V^*$. So $\dim \wedge^k V^* = C^n_k$. For $k > n$, we will have at least two of the e^*'s in the wedge product to be the same, so all possible wedge products will become 0. Therefore $\wedge^k V^* = \{0\}$. ■

Exercise 3.34.

$$\begin{aligned}
&(e^*_{i_1} \wedge ... \wedge e^*_{i_k})(e_{j_1}, ..., e_{j_k}) \\
&= \sum_{\sigma \in S_k} sgn(\sigma)(e^*_{\sigma(i_1)} \otimes ... \otimes e^*_{\sigma(i_k)})(e_{j_1}, ..., e_{j_k}) \\
&= \sum_{\sigma \in S_k} sgn(\sigma) e^*_{\sigma(i_1)}(e_{j_1}) ... e^*_{\sigma(i_k)}(e_{j_k}) \\
&= \sum_{\sigma \in S_k} sgn(\sigma) \delta_{\sigma(i_1) j_1} ... \delta_{\sigma(i_k) j_k}
\end{aligned}$$

Only one term in this summation can survive because if there is a permutation $\sigma' \in S_k$ such that $\sigma(i_m) = j_m$ for all $1 \leq m \leq k$, then any other permutation will make at least one of the Kronecker delta zero. So we have

$$(e^*_{i_1} \wedge ... \wedge e^*_{i_k})(e_{j_1}, ..., e_{j_k}) = sgn(\sigma')\delta_{\sigma'(i_1)j_1}...\delta_{\sigma'(i_k)j_k}$$

Furthermore, if $\sigma(i_m) = j_m$ for any $1 \leq m \leq k$, the only possibility is that σ is the identity, and so $sgn(\sigma) = 1$. Therefore, we have

$$(e^*_{i_1} \wedge ... \wedge e^*_{i_k})(e_{j_1}, ..., e_{j_k}) = \delta_{i_1 j_1}...\delta_{i_k j_k}$$

$\blacksquare$

Exercise 3.35.

Let's rewrite the statement:

$$\omega_1, ..., \omega_n \in V^* \text{ are linearly dependent} \iff \omega_1 \wedge \omega_2 ... \wedge \omega_n = 0$$

Solution: $\omega_1, ..., \omega_n \in V^*$ are linearly dependent
$\iff$ one of $\omega_1, ..., \omega_n$ is a linear combination of the others
$\iff$ WLOG,say $\exists c_1, ...c_{n-1} \neq 0$ s.t. $w_n = c_1\omega_1 + ... + c_{n-1}\omega_{n-1}$
$\Rightarrow \omega_1 \wedge \omega_2 \wedge ...\omega_{n-1} \wedge \omega_n = \sum_{i=1}^{n-1} \omega_1 \wedge \omega_2 \wedge ...\omega_{n-1} \wedge (c_i\omega_i) = \sum_{i=1}^{n-1} c_i\omega_1 \wedge \omega_2 \wedge ...\omega_{n-1} \wedge \omega_i = 0$
As For $i = 1, 2, 3, ...n-1$, $\omega_i \wedge \omega_i = 0$
For $\omega_1 \wedge \omega_2 \wedge ...\omega_{n-1} \wedge \omega_n = 0$
$\Rightarrow \sum_{i_1=1}^n a_{1i_1} e^*_{i1} \wedge \sum_{i_2=1}^n a_{2i_2} e^*_{i2} \wedge ... \wedge \sum_{i_n=1}^n a_{ni_n} e^*_{in} = 0$
$\Rightarrow \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} ... a_{n\sigma(n)} e^*_1 \wedge e^*_2 \wedge ... \wedge e^*_n = 0$
$\Rightarrow det(A) e^*_1 \wedge e^*_2 \wedge ... \wedge e^*_n = 0$
where A= $\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix}$
$\Rightarrow det(A) = 0 \Rightarrow det(A^T) = 0 As e^*_1 \wedge e^*_2 \wedge ... \wedge e^*_n \neq 0$
$\Rightarrow$Columns of A^T are linearly dependent.
$\Rightarrow$Say $\exists c_1, ...c_{n-1} \neq 0$ s.t.
$(a_{n1}, a_{n2}, a_{n3}, ..., a_{nn}) = c_1(a_{11}, a_{12}, a_{13}, ..., a_{1n}) + ... + c_{n-1}(a_{(n-1)1}, a_{(n-1)_2}, a_{(n-1)3}, ..., a_{(n-1)n})$
$\Rightarrow \exists c_1, ...c_{n-1} \neq 0$ s.t. $w_n = c_1\omega_1 + ... + c_{n-1}\omega_{n-1}$ $\blacksquare$

Exercise 3.36.

$$\begin{aligned} \omega_1 \wedge \cdots \omega_k &= \sum_{\text{distinct } j_1, \cdots, j_k} a_{1j_1} a_{2j_2} \cdots a_{kj_k} e^*_{j_1} \wedge \cdots \wedge e^*_{j_k} \\ &= \sum_{\{j'_1, \cdots, j'_k\} \in X_k} \sum_{\sigma \in S_k} a_{1\sigma(j'_1)} \cdots a_{1\sigma(j'_k)} e^*_{\sigma(j'_1)} \wedge \cdots \wedge e^*_{\sigma(j'_k)} \\ &= \sum_{\{j'_1, \cdots, j'_k\} \in X_k} \left(\sum_{\sigma \in S_k} \text{sgn}(\sigma) a_{1\sigma(j'_1)} \cdots a_{1\sigma(j'_k)} \right) e^*_{j'_1} \wedge \cdots \wedge e^*_{j'_k} \end{aligned}$$

where X_k is the set of subsets of $\{1, \cdots, n\}$ with k elements and define as convention that $j'_1 < \cdots < j'_k$. $\blacksquare$

Exercise 3.37. For any $(v_1, ..., v_n) \in \mathbb{R}^n \times ... \times \mathbb{R}^n$,

$$\begin{aligned}
&e_1^* \wedge ... \wedge e_n^*(v_1, ..., v_n)\\
&= \sum_{\sigma \in S_n} sgn(\sigma) e_{\sigma(1)}^* \otimes ... \otimes e_{\sigma(n)}^*(v_1, ..., v_n)\\
&= \sum_{\sigma \in S_n} sgn(\sigma) e_{\sigma(1)}^* \otimes ... \otimes e_{\sigma(n)}^*(\sum_{i=1}^{n} v_{1i} e_i, ..., \sum_{i=1} v_{ni} e_i)\\
&= \sum_{\sigma \in S_n} sgn(\sigma) e_{\sigma(1)}^*(\sum_{i=1}^{n} v_{1i} e_i) ... e_{\sigma(n)}^*(\sum_{i=1} v_{ni} e_i)\\
&= \sum_{\sigma \in S_n} sgn(\sigma)(\sum_{i=1}^{n} v_{1i} e_{\sigma(1)}^*(e_i)) ... (\sum_{i=1} v_{ni} e_{\sigma(n)}^*(e_i))\\
&= \sum_{\sigma \in S_n} sgn(\sigma)(\sum_{i=1}^{n} v_{1i} \delta_{\sigma(1)i}) ... (\sum_{i=1} v_{ni} \delta_{\sigma(n)i})\\
&= \sum_{\sigma \in S_n} sgn(\sigma) v_{1\sigma(1)} ... v_{n\sigma(n)}\\
&= det(v_1, ..., v_n)\\
\Rightarrow &e_1^* \wedge ... \wedge e_n^* = \det
\end{aligned}$$

■

Exercise 3.39.

$$\begin{aligned}
\omega &= \sum_{i=1}^{n} \omega_{2i-1} \wedge \omega_{2i}\\
\omega^2 &:= \omega \wedge \omega = 2 \sum_{1 \le i_1 < i_2 \le n} \omega_{2i_1 - 1} \wedge \omega_{2i_1} \wedge \omega_{2i_2 - 1} \wedge \omega_{2i_2}\\
\omega^3 &:= \omega \wedge \omega \wedge \omega = 2 * 3 \sum_{1 \le i_1 < i_2 < i_3 \le n} \omega_{2i_1 - 1} \wedge \omega_{2i_1} \wedge \omega_{2i_2 - 1} \wedge \omega_{2i_2} \wedge \omega_{2i_3 - 1} \wedge \omega_{2i_3}
\end{aligned}$$

Inductively,

$$\omega^k = 2 * 3 * ... * k \sum_{1 \le i_1 < i_2 < ... < i_k \le n} \omega_{2i_1 - 1} \wedge \omega_{2i_1} \wedge \omega_{2i_2 - 1} \wedge \omega_{2i_2} \wedge ... \wedge \omega_{2i_k - 1} \wedge \omega_{2i_k}$$

Thus

$$\begin{aligned}
\omega^n &= \omega \wedge \omega \wedge ... \wedge \omega = (2 * 3 * ... * n) \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 \wedge ... \wedge \omega_{2n-1} \wedge \omega_{2n}\\
&= n! \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 \wedge ... \wedge \omega_{2n-1} \wedge \omega_{2n}
\end{aligned}$$

after taking vedge n times. ■

Exercise 3.40.

$$\begin{aligned}
&du^1 \wedge du^2 \wedge ... \wedge du^n \\
&= (\sum_{i_1=1}^{n} \frac{\partial u^1}{\partial v^{i_1}} dv^{i_1}) \wedge (\sum_{i_2=1}^{n} \frac{\partial u^2}{\partial v^{i_2}} dv^{i_2}) \wedge ... \wedge (\sum_{i_n=1}^{n} \frac{\partial u^n}{\partial v^{i_n}} dv^{i_n}) \\
&= \sum_{i_1, i_2, ..., i_n \text{distinct}} \frac{\partial u^1}{\partial v^{i_1}} \frac{\partial u^2}{\partial v^{i_2}} ... \frac{\partial u^n}{\partial v^{i_n}} dv^{i_1} \wedge dv^{i_2} \wedge ... \wedge dv^{i_n} \\
&= \sum_{\sigma \in S_n} \frac{\partial u^1}{\partial v^{\sigma(1)}} \frac{\partial u^2}{\partial v^{\sigma(2)}} ... \frac{\partial u^n}{\partial v^{\sigma(n)}} dv^{\sigma(1)} \wedge dv^{\sigma(2)} \wedge ... \wedge dv^{\sigma(n)} \\
&= (\sum_{\sigma \in S_n} sgn(\sigma) \frac{\partial u^1}{\partial v^{\sigma(1)}} \frac{\partial u^2}{\partial v^{\sigma(2)}} ... \frac{\partial u^n}{\partial v^{\sigma(n)}}) dv^1 \wedge dv^2 \wedge ... \wedge dv^n \\
&= det \frac{\partial(u_1, ..., u_n)}{\partial(v_1, ..., v_n)} dv^1 \wedge dv^2 \wedge ... \wedge dv^n
\end{aligned}$$

■

Exercise 3.41.

($\Rightarrow$) If $T \in \wedge^2(\mathbb{R}^2)^*$, then T will be of the form $T = dx^1 \otimes dx^2 - dx^2 \otimes dx^1$. So for any $v \in \mathbb{R}^2$,

$$T(v, v) = dx^1(v)dx^2(v) - dx^2(v)dx^1(v) = 0$$

($\Leftarrow$) Suppose for any $v \in \mathbb{R}^3$, $T(v, v) = 0$ and $T \notin \wedge^2(\mathbb{R}^2)^*$, then T will be of the form

(i) $T = \pm dx^1 \otimes dx^2$

(ii) $T = \pm dx^1 \otimes dx^1$

(iii) $T = \pm(dx^1 \otimes dx^2 + dx^2 \otimes dx^1)$

(iv) $T = \pm(dx^1 \otimes dx^1 + dx^2 \otimes dx^2)$

For case (i), if we pick $v = a\partial_{x_1} + b\partial_{x_2}$ $(a, b \neq 0)$, then $T(v, v) = (\pm)ab \neq 0$. The argument for (ii) is similar.
For case (iii), if we pick the same v as in case (i), we will have $T(v, v) = (\pm)(2ab) \neq 0$. The argument for case (iv) will be similar.
Therefore, in any cases of $T \notin \wedge^2(\mathbb{R}^2)^*$ will lead to a contradiction, so $T \in \wedge^2(\mathbb{R}^2)^*$.

■

Exercise 3.42.

$$\begin{aligned}
\omega &= \sum_{j_1, j_2, ... j_k=1}^{n} \omega_{j_1 j_2 ... j_k} du^{j_1} \wedge du^{j_2} \wedge ... du^{j_k} \\
&= \sum_{j_1, j_2, ... j_k=1}^{n} \omega_{j_1 j_2 ... j_k} (\sum_{i_1=1}^{n} \frac{\partial u^{j_1}}{\partial v^{i_1}} dv^{i_1}) \wedge (\sum_{i_2=1}^{n} \frac{\partial u^{j_2}}{\partial v^{i_2}} dv^{i_2}) \wedge ... \wedge (\sum_{i_k=1}^{n} \frac{\partial u^{j_k}}{\partial v^{i_k}} dv^{i_k}) \\
&= \sum_{i_1, i_2, ... i_k=1}^{n} (\sum_{j_1, j_2, ... j_k=1}^{n} \frac{\partial u^{j_1}}{\partial v^{i_1}} ... \frac{\partial u^{j_k}}{\partial v^{i_k}} \omega_{j_1 j_2 ... j_k}) dv^{i_1} \wedge dv^{i_2} \wedge ... \wedge dv^{i_k} \\
&= \sum_{i_1, i_2, ... i_k=1}^{n} \omega'_{i_1 i_2 ... i_k} dv^{i_1} \wedge dv^{i_2} \wedge ... \wedge dv^{i_k}
\end{aligned}$$

where $\omega'_{i_1 i_2 \dots i_k} = \sum_{j_1, j_2, \dots j_k=1}^{n} \frac{\partial u^{j_1}}{\partial v^{i_1}} \cdots \frac{\partial u^{j_k}}{\partial v^{i_k}} \omega_{j_1 j_2 \dots j_k}$.

For $\omega = \sum_{j_1, j_2, \dots j_k=1}^{n} \omega_{j_1 j_2 \dots j_k} du^{j_1} \wedge du^{j_2} \wedge \dots du^{j_k}$,

$$
\begin{aligned}
d\omega &= \sum_{j_1, j_2, \dots j_k=1}^{n} (\sum_{j=1}^{n} \frac{\partial \omega_{j_1 j_2 \dots j_k}}{\partial u^j} du^j) \wedge du^{j_1} \wedge du^{j_2} \wedge \dots du^{j_k} \\
&= \sum_{j_1, j_2, \dots j_k=1}^{n} (\sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial \omega_{j_1 j_2 \dots j_k}}{\partial v^i} \frac{\partial v^i}{\partial u^j} du^j) \\
&\qquad \wedge (\sum_{i_1=1}^{n} \frac{\partial u^{j_1}}{\partial v^{i_1}} dv^{i_1}) \wedge (\sum_{i_2=1}^{n} \frac{\partial u^{j_2}}{\partial v^{i_2}} dv^{i_2}) \wedge \dots \wedge (\sum_{i_k=1}^{n} \frac{\partial u^{j_k}}{\partial v^{i_k}} dv^{i_k}) \\
&= \sum_{j_1, j_2, \dots j_k=1}^{n} (\sum_{i=1}^{n} \frac{\partial \omega_{j_1 j_2 \dots j_k}}{\partial v^i} \sum_{j=1}^{n} \frac{\partial v^i}{\partial u^j} du^j) \\
&\qquad \wedge (\sum_{i_1=1}^{n} \frac{\partial u^{j_1}}{\partial v^{i_1}} dv^{i_1}) \wedge (\sum_{i_2=1}^{n} \frac{\partial u^{j_2}}{\partial v^{i_2}} dv^{i_2}) \wedge \dots \wedge (\sum_{i_k=1}^{n} \frac{\partial u^{j_k}}{\partial v^{i_k}} dv^{i_k}) \\
&= \sum_{j_1, j_2, \dots j_k=1}^{n} (\sum_{i=1}^{n} \frac{\partial \omega_{j_1 j_2 \dots j_k}}{\partial v^i} dv^i) \wedge (\sum_{i_1=1}^{n} \frac{\partial u^{j_1}}{\partial v^{i_1}} dv^{i_1}) \wedge (\sum_{i_2=1}^{n} \frac{\partial u^{j_2}}{\partial v^{i_2}} dv^{i_2}) \wedge \dots \wedge (\sum_{i_k=1}^{n} \frac{\partial u^{j_k}}{\partial v^{i_k}} dv^{i_k}) \\
&= \sum_{i_1, i_2, \dots i_k=1}^{n} (\sum_{i=1}^{n} \sum_{j_1, j_2, \dots j_k=1}^{n} \frac{\partial u^{j_1}}{\partial v^{i_1}} \cdots \frac{\partial u^{j_k}}{\partial v^{i_k}} \frac{\partial \omega_{j_1 j_2 \dots j_k}}{\partial v^i} dv^i) \wedge dv^{i_1} \wedge dv^{i_2} \wedge \dots \wedge dv^{i_k} \\
&= \sum_{i_1, i_2, \dots i_k=1}^{n} (\sum_{i=1}^{n} \frac{\partial \omega'_{i_1 i_2 \dots i_k}}{\partial v^i} dv^i) \wedge dv^{i_1} \wedge dv^{i_2} \wedge \dots \wedge dv^{i_k}
\end{aligned}
$$

■

Exercise 3.43.

$$
\begin{aligned}
d\omega &= (\frac{\partial F_1}{\partial x} dx + \frac{\partial F_1}{\partial y} dy + \frac{\partial F_1}{\partial z} dz) \wedge dy \wedge dz \\
&\qquad + (\frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz) \wedge dz \wedge dx + (\frac{\partial F_3}{\partial x} dx + \frac{\partial F_3}{\partial y} dy + \frac{\partial F_3}{\partial z} dz) \wedge dx \wedge dy \\
&= \frac{\partial F_1}{\partial x} dx \wedge dy \wedge dz + \frac{\partial F_2}{\partial y} dy \wedge dz \wedge dx + \frac{\partial F_3}{\partial z} dz \wedge dx \wedge dy \\
&= \frac{\partial F_1}{\partial x} dx \wedge dy \wedge dz + \frac{\partial F_2}{\partial y} dx \wedge dy \wedge dz + \frac{\partial F_3}{\partial z} dx \wedge dy \wedge dz \\
&= (\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}) dx \wedge dy \wedge dz
\end{aligned}
$$

Consider $F = (F_1, F_2, F_3)$ be a $\mathbb{R}^3$ vector field. As div(F) $= \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$
In this case, d is analog with the divergence operator in Multivariable Calculus. ■

Exercise 3.44.

$$\begin{aligned}\omega \wedge \eta &= (x\ dx - y\ dy) \wedge (z\ dx \wedge dy + x\ dy \wedge dz)\\
&= xz\ dx \wedge dx \wedge dy - yz\ dy \wedge dx \wedge dy + x^2\ dx \wedge dy \wedge dz - xy\ dy \wedge dy \wedge dz\\
&= 0 + yz\ dx \wedge dy \wedge dy + x^2\ dx \wedge dy \wedge dz - 0\\
&= x^2\ dx \wedge dy \wedge dz\end{aligned}$$

$$\begin{aligned}\omega \wedge \eta \wedge \theta &= (x^2\ dx \wedge dy \wedge dz) \wedge (z\ dy)\\
&= x^2 z\ dx \wedge dy \wedge dz \wedge dy\\
&= -x^2 z\ dx \wedge dy \wedge dy \wedge dz\\
&= 0\end{aligned}$$

$$d\omega = \Big(\frac{\partial x}{\partial x}dx + \frac{\partial x}{\partial y}dy + \frac{\partial x}{\partial z}dz\Big) \wedge dx - \Big(\frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial y}dy + \frac{\partial y}{\partial z}dz\Big) \wedge dy = 0$$

$$\begin{aligned}d\eta &= \Big(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy + \frac{\partial z}{\partial z}dz\Big) \wedge dx \wedge dy + \Big(\frac{\partial x}{\partial x}dx + \frac{\partial x}{\partial y}dy + \frac{\partial x}{\partial z}dz\Big) \wedge dy \wedge dz\\
&= dz \wedge dx \wedge dy + dx \wedge dy \wedge dz\\
&= 2\ dx \wedge dy \wedge dz\end{aligned}$$

$$d\theta = \Big(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy + \frac{\partial z}{\partial z}dz\Big) \wedge dy = -dy \wedge dz$$

■

Exercise 3.45.

Consider the local coordinates $(u_1, ..., u_n)$ and let

$$\alpha = \sum_{i_1,...,i_k=1}^{n} \alpha_{i_1 \dots i_k} du^{i_1} \wedge ... \wedge du^{i_k}$$

$$\beta = \sum_{j_1,...,j_r=1}^{n} \beta_{j_1 \dots j_r} du^{j_1} \wedge ... \wedge du^{j_r}$$

Then

$$\alpha \wedge \beta = \sum_{i_1,...,i_k,j_1,...,j_r=1}^{n} \alpha_{i_1 \dots i_k}\beta_{j_1 \dots j_r} du^{i_1} \wedge ... \wedge du^{i_k} \wedge du^{j_1} \wedge ... \wedge du^{j_r}$$

So

$$
\begin{aligned}
d(\alpha\wedge\beta) = & \sum_{i_1,\dots,i_k,j_1,\dots,j_r=1}^{n}\sum_{i=1}^{n}\frac{\partial(\alpha_{i_1\dots i_k}\beta_{j_1\dots j_r})}{\partial u_i}du^i\wedge du^{i_1}\wedge\dots\wedge du^{i_k}\wedge du^{j_1}\wedge\dots\wedge du^{j_r}\\
= & \sum_{i_1,\dots,i_k,j_1,\dots,j_r=1}^{n}\sum_{i=1}^{n}\Big(\frac{\partial\alpha_{i_1\dots i_k}}{\partial u_i}\beta_{j_1\dots j_r}+\alpha_{i_1\dots i_k}\frac{\partial\beta_{j_1\dots j_r}}{\partial u_i}\Big)du^i\wedge du^{i_1}\wedge\\
& \dots\wedge du^{i_k}\wedge du^{j_1}\wedge\dots\wedge du^{j_r}\\
= & \sum_{i_1,\dots,i_k,j_1,\dots,j_r=1}^{n}\sum_{i=1}^{n}\frac{\partial\alpha_{i_1\dots i_k}}{\partial u_i}\beta_{j_1\dots j_r}du^i\wedge du^{i_1}\wedge\dots\wedge du^{i_k}\wedge du^{j_1}\wedge\dots\wedge du^{j_r}\\
& +\sum_{i_1,\dots,i_k,j_1,\dots,j_r=1}^{n}\sum_{i=1}^{n}\alpha_{i_1\dots i_k}\frac{\partial\beta_{j_1\dots j_r}}{\partial u_i}(-1)^k du^{i_1}\wedge\dots\wedge du^{i_k}\wedge du^i\wedge du^{j_1}\wedge\dots\wedge du^{j_r}\\
= & \Big(\sum_{i_1,\dots,i_k=1}^{n}\sum_{i=1}^{n}\frac{\partial\alpha_{i_1\dots i_k}}{\partial u_i}du^i\wedge du^{i_1}\wedge\dots\wedge du^{i_k}\Big)\wedge\Big(\sum_{j_1,\dots,j_r=1}^{n}\beta_{j_1\dots j_r}du^{j_1}\wedge\dots\wedge du^{j_r}\Big)\\
& +(-1)^k\Big(\sum_{i_1,\dots,i_k=1}^{n}\sum_{i=1}^{n}\alpha_{i_1\dots i_k}du^{i_1}\wedge\dots\wedge du^{i_k}\Big)\wedge\Big(\sum_{j_1,\dots,j_r=1}^{n}\sum_{i=1}^{n}\frac{\partial\beta_{j_1\dots j_r}}{\partial u_i}du^i\wedge du^{j_1}\wedge\dots\wedge du^{j_r}\Big)\\
= & \, d\alpha\wedge\beta+(-1)^k\alpha\wedge d\beta
\end{aligned}
$$

There is $(-1)^k$ but no $(-1)^r$ involved in the result because we need to switch the du^i to the right k times in order to get $d\beta$ in the second term, but we don't need to switch it to get $d\alpha$ in the first term. ■

Exercise 3.46.

WLOG, Assume $\alpha\in\wedge^k T^*M,\beta\in\wedge^r T^*M,\gamma\in\wedge^s T^*M$

$$
\begin{aligned}
d(\alpha\wedge\beta\wedge\gamma) &= d((\alpha\wedge\beta)\wedge\gamma)\\
&= d(\alpha\wedge\beta)\wedge\gamma+(-1)^{k+r}(\alpha\wedge\beta)\wedge d\gamma\\
&= (d\alpha\wedge\beta+(-1)^k\alpha\wedge d\beta)\wedge\gamma+(-1)^{k+r}(\alpha\wedge\beta)\wedge 0\\
&= (0\wedge\beta+(-1)^k\alpha\wedge 0)\wedge\gamma+0\\
&= 0
\end{aligned}
$$

(Argument above is used By proposition 3.35) ■

Exercise 3.47.

$d\omega = dx\wedge dx - dy\wedge dy = 0$ by definition of wedge. Meanwhile, $d(x^2/2-y^2/2) = xdx-ydy=\omega$, so ω is closed and exact.

$d\eta = zdx\wedge dy + xdy\wedge dz\neq 0$, so it is not closed. Since exact imply closed, not closed imply not exact, so η is not exact and not closed.

$d\theta = dz\wedge dy\neq 0$, so it is not closed. Since exact imply closed, not closed imply not exact, so θ is not exact and not closed. ■

Exercise 3.49.

$$\begin{aligned}
F(\theta,\varphi) &= (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta) \\
\omega &= x dy \otimes dz \\
\frac{\partial y}{\partial\theta} &= -\sin\theta\sin\varphi \\
\frac{\partial y}{\partial\varphi} &= \cos\theta\cos\varphi \\
\frac{\partial z}{\partial\theta} &= \cos\theta \\
\frac{\partial z}{\partial\varphi} &= 0 \\
\therefore \imath^*\omega &= \cos\theta\cos\varphi(\imath^* dy)\otimes(\imath^* dz) \\
&= (\cos\theta\cos\varphi)(-\sin\theta\sin\varphi d\theta + \cos\theta\cos\varphi d\varphi)\otimes(\cos\theta d\theta) \\
&= -\sin\theta\cos^2\theta\sin\varphi\cos\varphi d\theta\otimes d\theta + \cos^3\theta\cos\varphi d\varphi\otimes d\theta
\end{aligned}$$

■

Exercise 3.50.

Suppose $\dim M = m$ and $\dim N = n$, then

$$\Phi^*(df) = \Phi^*\left(\sum_{j=1}^{n}\frac{\partial f}{\partial v_j}dv^j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\partial f}{\partial v_j}\frac{\partial v_j}{\partial u_i}du^i$$

meanwhile, following the chain rule,

$$d(\Phi^* f) = \sum_{i=1}^{m}\frac{\partial(f\circ\Phi)}{\partial u_i}du^i = \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\partial f}{\partial v_j}\frac{\partial v_j}{\partial u_i}du^i$$

the expressions are identical, this proved $\Phi^*(df) = d(\Phi^* f)$. ■

Exercise 3.51.

Suppose $\{\mathcal{U}_\alpha\}, \{\mathcal{V}_\beta\}, \{\mathcal{W}_\gamma\}$ are the domain of the parametrization of M^m, N^n and P^p respectively. Suppose the concerned expressions act of the cotangent vector dw^i, we have

$$(\Psi\circ\Phi)^* dw^i = \sum_{j=1}^{m}\frac{\partial w_i}{\partial u_j}du^j = \sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\partial w_i}{\partial v_k}\frac{\partial v_k}{\partial u_j}du^j$$

where the second equality in the above follows from chain rule. Meanwhile,

$$\Phi^*\circ\Psi^*(dw^i) = \Phi^*\left(\sum_{k=1}^{n}\frac{\partial w_i}{\partial v_k}dv^k\right) = \sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\partial w_i}{\partial v_k}\frac{\partial v_k}{\partial u_j}du^j$$

since the both $\Phi^*\circ\Psi^*$ and $(\Psi\circ\Phi)^*$ give the same expression when acting on the cotangent vector dw^i, we conclude $\Phi^*\circ\Psi^* = (\Psi\circ\Phi)^*$. ■

Exercise 3.53.

We make use of Proposition 3.50.
If ω is closed, then $d\omega = 0$, so $d(\Phi^*\omega) = \Phi^*(d\omega) = 0$ which is closed.
If ω is exact, then $\omega = d\eta$, thus $\Phi^*\omega = \Phi^*(d\eta) = d(\Phi^*\eta)$ which is exact. ■