Problem Set #1

MATH 4033, Calculus on Manifold, Spring 2019

Problem 2. *Consider a regular surface Σ in $\mathbb{R}^3$. Denote $\nu : \Sigma \to \mathbb{R}^3$ be a smooth map of unit normal vectors to Σ. Let $f : \Sigma \to (0, \infty)$ be a C^∞ function on Σ, and consider the set:*

$$\hat{\Sigma} := \{p + f(p)\nu(p) : p \in \Sigma\}.$$

(a) Suppose $F(u_1, u_2)$ is a C^∞ local parametrization of Σ.

i. Show that $\dfrac{\partial \nu}{\partial u_i}(p) \in T_p\Sigma$.

Solution. By definition, $\nu(p)$ is of unit length for any $p \in \Sigma$; that is,

$$\langle \nu(p), \nu(p) \rangle = 1.$$

Differentiating both sides with respect to u_i using the product rule, we obtain

$$\begin{aligned} \left\langle \frac{\partial \nu}{\partial u_i}(p), \nu(p) \right\rangle + \left\langle \nu(p), \frac{\partial \nu}{\partial u_i}(p) \right\rangle &= 0 \\ \therefore \qquad \left\langle \frac{\partial \nu}{\partial u_i}(p), \nu(p) \right\rangle &= 0 \end{aligned}$$

Thus $\dfrac{\partial \nu}{\partial u_i}(p)$ is orthogonal to the normal vector $\nu(p)$ at p. Hence, $\dfrac{\partial \nu}{\partial u_i}(p)$ is in the tangent plane $T_p\Sigma$. □

ii. Consider the linear map $h : T_p\Sigma \to T_p\Sigma$ defined by:

$$h\left(\frac{\partial F}{\partial u_i}\right) := \frac{\partial \nu}{\partial u_i}$$

and extends linearly to all of $T_p\Sigma$. Show that h is self-adjoint with respect to the standard dot product, i.e.

$$\langle h(X), Y \rangle = \langle X, h(Y) \rangle \quad \textit{for any } X, Y \in T_p\Sigma.$$

Solution. It suffices to show that the equality holds for the basis vectors $\dfrac{\partial F}{\partial u_i}(p)$, since h is linear and the inner product is bilinear.

Since ν is orthogonal to the tangent vectors, i.e.

$$\left\langle \nu, \frac{\partial F}{\partial u_i} \right\rangle = 0$$

for any i, thus we have

$$\frac{\partial}{\partial u_i}\left\langle \nu, \frac{\partial F}{\partial u_j}\right\rangle = 0 = \frac{\partial}{\partial u_j}\left\langle \nu, \frac{\partial F}{\partial u_i}\right\rangle$$

for any i, j. Expanding both sides and using the definition of h, we have

$$\left\langle \frac{\partial \nu}{\partial u_i}, \frac{\partial F}{\partial u_j}\right\rangle + \cancel{\left\langle \nu, \frac{\partial^2 F}{\partial u_i \partial u_j}\right\rangle} = \left\langle \frac{\partial \nu}{\partial u_j}, \frac{\partial F}{\partial u_i}\right\rangle + \cancel{\left\langle \nu, \frac{\partial^2 F}{\partial u_j \partial u_i}\right\rangle}$$
$$\therefore \qquad \left\langle h\left(\frac{\partial F}{\partial u_i}\right), \frac{\partial F}{\partial u_j}\right\rangle = \left\langle \frac{\partial F}{\partial u_i}, h\left(\frac{\partial \nu}{\partial u_j}\right)\right\rangle$$

Thus the equality holds for the basis tangent vectors. By linearity, it holds for every $X, Y \in T_p\Sigma$. Hence h is self-adjoint. $\square$

(b) *From (a), we denote* $h\left(\frac{\partial F}{\partial u_i}\right) = \sum_j h_i^j \frac{\partial F}{\partial u_j}$. *Consider the map* $\hat{F}$ *defined on the same domain as* F*:*

$$\hat{F}(u_1, u_2) := F(u_1, u_2) + f(F(u_1, u_2))\nu(F(u_1, u_2)).$$

Suppose $\hat{F}$ *is a homeomorphism onto its image. Show that if the linear map:*

$$\mathrm{id} + fh : T_p\Sigma \to T_p\Sigma$$

is invertible for any $p \in \Sigma$*, then* $\hat{F}$ *is a* C^∞ *local parametrization of* $\hat{\Sigma}$*.*

Solution. Condition (1): Since f, F, ν are all smooth by assumption, thus $f \circ F$ and $\nu \circ F$ are smooth. Hence $\hat{F} = F + (f \circ F)(\nu \circ F)$ is smooth.

Condition (2): By assumption, $\hat{F}$ is already a homeomorphism onto its image.

Condition (3): We compute that:

$$\frac{\partial \hat{F}}{\partial u_i} = \frac{\partial F}{\partial u_i} + \frac{\partial (f \circ F)}{\partial u_i}\nu(u_1, u_2) + f(F(u_1, u_2))\frac{\partial \nu}{\partial u_i}$$

for any i. Using the definition of h, we can further write

$$\begin{aligned}\frac{\partial \hat{F}}{\partial u_i} &= \mathrm{id}\left(\frac{\partial F}{\partial u_i}\right) + \frac{\partial (f \circ F)}{\partial u_i}\nu(u_1, u_2) + f(F(u_1, u_2))h\left(\frac{\partial F}{\partial u_i}\right) \\ &= \underbrace{(\mathrm{id} + fh)\left(\frac{\partial F}{\partial u_i}\right)}_{\text{call this } X_i} + \frac{\partial (f \circ F)}{\partial u_i}\nu \\ &= X_i + \frac{\partial f}{\partial u_i}\nu\end{aligned}$$

for any i. Thus

$$\begin{aligned}\frac{\partial \hat{F}}{\partial u_1} \times \frac{\partial \hat{F}}{\partial u_2} &= X_1 \times X_2 + \frac{\partial f}{\partial u_2} X_1 \times \nu + \frac{\partial f}{\partial u_1} \nu \times X_2 + \frac{\partial f}{\partial u_1}\frac{\partial f}{\partial u_2} \underbrace{\nu \times \nu}_{=0} \\ &= X_1 \times X_2 + \frac{\partial f}{\partial u_2} X_1 \times \nu + \frac{\partial f}{\partial u_1} \nu \times X_2\end{aligned}$$

Note that since F is a smooth local parametrization, thus $\frac{\partial F}{\partial u_i}$ and $\frac{\partial F}{\partial u_2}$ are linearly independent. Now, if the linear map id$+fh$ is invertible, by linear algebra it must map linearly independent vectors to linearly independent vectors. Thus, X_1 and X_2 are linear independent, therefore the cross product $X_1 \times X_2$ is nonzero. Since it is parallel to ν, we have $\nu \cdot (X_1 \times X_2) \neq 0$.

On the other hand, $\nu \cdot (X_1 \times \nu) = \nu \cdot (\nu \times X_2) = 0$. Thus

$$\nu \cdot \left(\frac{\partial \hat{F}}{\partial u_1} \times \frac{\partial \hat{F}}{\partial u_2} \right) = \nu \cdot (X_1 \times X_2) \neq 0$$

Therefore it must follow that $\frac{\partial \hat{F}}{\partial u_1} \times \frac{\partial \hat{F}}{\partial u_2} \neq 0$. Hence condition (3) is satisfied.

Hence by definition, $\hat{F}$ is a smooth local parametrization of $\hat{\Sigma}$. $\square$