PROBLEM ONE = HW 3, Q3

a) Under the local coordinates $\overset{u_1, u_2, u_3}{(s, \varphi, \theta)}$, components of X and Y are both constant functions (1 or 0), hence by (3.11) of P.74, we have:

$$\mathcal{L}_X \underset{\substack{\uparrow \\ \text{1-form}}}{\alpha} = \sum_{i,j} X^i \frac{\partial \alpha_j}{\partial u_i} du^j = \sum_j X(\alpha_j)\, du^j$$

$$\Rightarrow \mathcal{L}_X(ds) = X(1)\, ds + X(0)\, d\varphi + X(0)\, d\theta = 0.$$

$$\mathcal{L}_X\left(\frac{s}{s^3+s-2} ds\right) = X\left(\frac{s}{s^3+s-2}\right) ds + X\left(\frac{s}{s^3+s-2}\right) d\varphi + X\left(\frac{s}{s^3+s-2}\right) d\theta$$

$$= 0 \qquad \left(\text{note } X = \frac{\partial}{\partial \theta}\right)$$

$$\Rightarrow \mathcal{L}_X\left(\frac{s}{s^3+s-2} ds \otimes ds\right) = \mathcal{L}_X\left(\frac{s}{s^3+s-2} ds\right) \otimes ds + \frac{s}{s^3+s-2} ds \otimes \mathcal{L}_X ds$$

Text

$$= 0 \otimes ds + \frac{s}{s^3+s-2} ds \otimes 0 = 0.$$

Similarly, $\mathcal{L}_{\frac{\partial}{\partial\theta}}(\underset{\substack{\uparrow \\ \text{constant}}}{1}\, d\varphi) = 0$ and $\mathcal{L}_{\frac{\partial}{\partial\theta}}(\underset{\substack{\uparrow \\ \text{indep. of } \theta}}{s^2}\, d\varphi) = 0$

$\mathcal{L}_{\frac{\partial}{\partial\theta}}(1\, d\theta) = 0$ and $\mathcal{L}_{\frac{\partial}{\partial\theta}}(\underbrace{s^2 \sin^2\varphi}_{\text{indep. of } \theta}\, d\theta) = 0$

$$\Rightarrow \mathcal{L}_{\frac{\partial}{\partial\theta}}(s^2\, d\varphi \otimes d\varphi) = 0$$

and $\mathcal{L}_{\frac{\partial}{\partial\theta}}(s^2 \sin^2\varphi\, d\theta \otimes d\theta) = 0.$

$$\therefore \boxed{\mathcal{L}_{\frac{\partial}{\partial\theta}} g = 0}$$

Similar for $\mathcal{L}_Y g = \mathcal{L}_{\frac{\partial}{\partial\varphi}} g$. The only non-zero term is $\mathcal{L}_{\frac{\partial}{\partial\varphi}}(s^2\sin^2\varphi\, d\theta)$ as the others have components indep. of φ.

$$\parallel$$

$$\frac{\partial}{\partial\varphi}(s^2\sin^2\varphi)\, d\theta = s^2\cos\varphi\, d\theta$$

$$\Rightarrow \mathcal{L}_{\frac{\partial}{\partial\varphi}}(s^2\sin^2\varphi\, d\theta\otimes d\theta) = \mathcal{L}_{\frac{\partial}{\partial\varphi}}(s^2\sin^2\varphi\, d\theta)\otimes d\theta + s^2\sin^2\varphi\, d\theta\otimes \cancel{\mathcal{L}_{\frac{\partial}{\partial\varphi}} d\theta}^{\,0}$$

$$= 2s^2\sin\varphi\cos\varphi\, d\theta\otimes d\theta.$$

$$\therefore \boxed{\mathcal{L}_{\frac{\partial}{\partial\varphi}} g = s^2\sin 2\varphi\; d\theta\otimes d\theta}$$

$(i_X\Omega)(v_1,v_2) := \Omega(X,v_1,v_2)$, so if v_1 or $v_2 \parallel X$, we have $(i_X\Omega)(v_1,v_2) = 0$. Now $X = \frac{\partial}{\partial\theta}$, so we only need to consider the output

$$(i_{\frac{\partial}{\partial\theta}}\Omega)\left(\frac{\partial}{\partial s},\frac{\partial}{\partial\varphi}\right) = \Omega\left(\frac{\partial}{\partial\theta},\frac{\partial}{\partial s},\frac{\partial}{\partial\varphi}\right)$$

$$= \sqrt{\det[g]}\; ds\wedge d\varphi\wedge d\theta\left(\frac{\partial}{\partial\theta},\frac{\partial}{\partial s},\frac{\partial}{\partial\varphi}\right)$$

$$= \sqrt{\det[g]}\; d\theta\wedge ds\wedge d\varphi\left(\frac{\partial}{\partial\theta},\frac{\partial}{\partial s},\frac{\partial}{\partial\varphi}\right)$$

$$= \sqrt{\det[g]}\cdot 1$$

$$\Rightarrow \boxed{i_{\frac{\partial}{\partial\theta}}\Omega = \sqrt{\det[g]}\; ds\wedge d\varphi}$$

Similarly, $(i_{\frac{\partial}{\partial\varphi}}\Omega)\left(\frac{\partial}{\partial s},\frac{\partial}{\partial\theta}\right) = \Omega\left(\frac{\partial}{\partial\varphi},\frac{\partial}{\partial s},\frac{\partial}{\partial\theta}\right)$

$$= \sqrt{\det[g]}\; ds\wedge d\varphi\wedge d\theta\left(\frac{\partial}{\partial\varphi},\frac{\partial}{\partial s},\frac{\partial}{\partial\theta}\right) = -\sqrt{\det[g]}$$

$$\therefore\; i_{\frac{\partial}{\partial\varphi}}\Omega = -\sqrt{\det[g]}\; ds\wedge d\theta = \sqrt{\det[g]}\; d\theta\wedge ds.$$

Ω is a differential form

$\Rightarrow$ we can use Cartan's magic formula to compute $\mathcal{L}_X\Omega$ and $\mathcal{L}_Y\Omega$.

Ω is a 3-form in a 3-dim space M.

$\Rightarrow$ $d\Omega$ is a 4-form which must be <u>zero</u> in M.

$$\therefore \boxed{\mathcal{L}_X\Omega} = i_X(\overset{0}{\cancel{d\Omega}}) + d(i_X\Omega)$$

$$= d\left(\sqrt{\det[g]}\, ds \wedge d\varphi\right)$$

$$= \left(\frac{\partial}{\partial\theta}\sqrt{\det[g]}\, d\theta\right)\wedge ds \wedge d\varphi$$

other terms vanish when $\wedge ds \wedge d\varphi$.

$$= \boxed{0} \text{ as } \det[g] \text{ is indep. of } \theta.$$

Similarly, $$\boxed{\mathcal{L}_Y\Omega} = d(i_Y\Omega) = d\left(\sqrt{\det[g]}\, d\theta \wedge ds\right)$$

$$= \left(\frac{\partial}{\partial\varphi}\sqrt{\det[g]}\, d\varphi\right)\wedge d\theta \wedge ds$$

$$= \boxed{\left(\frac{s^5}{s^3+s-2}\right)^{\frac{1}{2}} \cos\varphi\, d\varphi \wedge d\theta \wedge ds}$$

(b) We will show $\exists$ a smooth function $f(r)$ such that the change of variable $s = f(r)$ will give the desired change-of-coordinates result. $\rightarrow ds = f'(r)\,dr$

$$g = \frac{s}{s^3+s-2} ds\otimes ds + s^2\left(d\varphi\otimes d\varphi + \sin^2\varphi\, d\theta\otimes d\theta\right)$$

$$= \frac{f(r)}{f(r)^3+f(r)-2} f'(r)^2\, dr\otimes dr + f(r)^2\left(d\varphi\otimes d\varphi + \sin^2\varphi\, d\theta\otimes d\theta\right)$$

To get the desired result, we need:

$$\frac{f(r)}{f(r)^3+f(r)-2} f'(r)^2 = 1$$

We claim that $\exists f(r)$ s.t.

$$f' = \sqrt{\frac{f^3+f-2}{f}} = \sqrt{\underbrace{f^2+1-\frac{2}{f}}}$$

>0 as long as $\underline{f>1}$.

Impose an initial condition

$$f(0) = 2$$

then the IVP : $f' = \sqrt{f^2+1-\frac{2}{f}}$, $f(0)=2$. $\exists$ a C^∞

solution $f(r)$, $r \in (-A, B)$, by <u>ODE existence</u>.

As $f'>0$, f is strictly increasing with C^∞ inverse.
(hence +ve) (by inverse function theorem)

The change-of-coordinates

$(s, \varphi, \theta) \to (r, \varphi, \theta)$ by the rule $s = f(r)$

gives the desired result.

PROBLEM 2 = HW2 Q1

(a) Suppose Σ is covered by a regular local parametrization $F(u,v): \mathcal{U} \to \Sigma$

one can then define an induced local parametrization $\tilde{F}: \mathcal{U} \times \mathbb{R} \to N\Sigma$ by:

$$\tilde{F}(u_1,u_2,t) := \Big(\underbrace{F(u_1,u_2)}_{\text{in } \Sigma},\ t \underbrace{\frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2}}_{\neq 0 \text{ by regular condition.}} \Big)$$

one-to-one since $\frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2} \neq 0$.

These $\tilde{F}$'s can certainly cover $N\Sigma$.

Now suppose $F(u_1,u_2)$, $G(v_1,v_2)$ are two overlapping parametrizations of Σ, then we compute the transition map $\tilde{G}^{-1} \circ \tilde{F}$:

$$\tilde{G}^{-1} \circ \tilde{F}(u_1,u_2,t) = \tilde{G}^{-1}\Big(F(u_1,u_2),\ t\frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2}\Big)$$

$$= \tilde{G}^{-1}\Big(F(u_1,u_2),\ t \det \frac{\partial(v_1,v_2)}{\partial(u_1,u_2)} \frac{\partial G}{\partial v_1} \times \frac{\partial G}{\partial v_2}\Big)$$

$$= \Big(\underbrace{G^{-1} \circ F(u_1,u_2)}_{C^\infty \text{ for regular surface}},\ \underbrace{t \det \frac{\partial(v_1,v_2)}{\partial(u_1,u_2)}}_{= t \det D(G^{-1}\circ F)} \Big)$$

$t \det D(G^{-1} \circ F)$ is C^∞ as $D(G^{-1}\circ F)$ has C^∞ entries.

$\therefore$ $\tilde{F}$ and $\tilde{G}$ have C^∞ transition maps.

$\Rightarrow$ $N\Sigma$ is a C^∞ 3-manifold.

(b) Given a local parametrization $F(u_1,u_2) : \mathcal{U} \to \Sigma$ of Σ, define an induced parametrization of $L\Sigma$ by

$$\overline{F}(u_1,u_2,t) : \mathcal{U}\times\mathbb{R} \longrightarrow L\Sigma$$

$$(u_1,u_2,t) \mapsto (\underbrace{F(u_1,u_2)}_{\text{in } \Sigma}, \underbrace{tF(u_1,u_2)}_{\text{in } \mathbb{R}^3}).$$

one-to-one since $\underbrace{F(u_1,u_2) \neq (0,0,0)}_{\text{not in } \Sigma}$

Given another local parametrization $G(v_1,v_2) : \mathcal{V} \to \Sigma$, and consider a similarly-defined $\overline{G} : \mathcal{V}\times\mathbb{R} \to L\Sigma$,

the transition map is given by:

$$\overline{G}^{-1}\circ\overline{F}(u_1,u_2,t) = \overline{G}^{-1}(F(u_1,u_2), tF(u_1,u_2)) = (\underbrace{G^{-1}\circ F(u_1,u_2)}_{(v_1,v_2)}, t)$$

$$\overline{G}(v_1,v_2,s) = (F(u_1,u_2), tF(u_1,u_2))$$

$$\Leftrightarrow \begin{cases} G(v_1,v_2) = F(u_1,u_2) \Leftrightarrow (v_1,v_2) = G^{-1}\circ F(u_1,u_2) \\ s = t \end{cases}$$

$G^{-1}\circ F$ is C^∞ $\Rightarrow$ $\overline{G}^{-1}\circ\overline{F}$ is C^∞.

$\therefore$ $L\Sigma$ is a 3-manifold (clearly that these $\overline{F}$'s cover the whole $L\Sigma$.)

To show $L\Sigma$ is a submanifold of $\Sigma\times\mathbb{R}^3$, we show $\iota : L\Sigma \to \Sigma\times\mathbb{R}^3$ is an immersion.

$$\begin{array}{ccc} L\Sigma & \xrightarrow{\iota} & \Sigma\times\mathbb{R}^3 \\ \overline{F}\nearrow & & \uparrow F\times \mathrm{id} \\ \mathcal{U}\times\mathbb{R} & & \mathcal{U}\times\mathbb{R}^3 \end{array}$$

express $F(u_1,u_2) = (x(u_1,u_2), y(u_1,u_2), z(u_1,u_2))$

then $(F\times\mathrm{id})^{-1}\circ\iota\circ\overline{F}(u_1,u_2,t) = (F\times\mathrm{id})^{-1}(F(u_1,u_2), tF(u_1,u_2)) = (u_1,u_2, tF(u_1,u_2))$

$$= (u_1,u_2, t\,x(u_1,u_2), t\,y(u_1,u_2), t\,z(u_1,u_2))$$

$$\therefore\ [\iota_*] = \frac{\partial(u_1,u_2,t\,x(u_1,u_2), t\,y(u_1,u_2), t\,z(u_1,u_2))}{\partial(u_1,u_2,t)}$$

$$= \underbrace{\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{*} & \begin{matrix} x \\ y \\ z \end{matrix} \end{array}\right]}_{}$$ ← linearly independent columns

As $(0,0,0) \notin \Sigma$, $(x,y,z) \neq (0,0,0)$ $\Rightarrow$ ι_* injective ☒

(c) Define the map $\Phi : N\Sigma \to L\Sigma$ by:

$$\Phi(\underset{\Sigma}{\underset{\uparrow}{p}}, \underset{N_p\Sigma}{\underset{\uparrow}{t\hat{n}(p)}}) := (\underset{\Sigma}{\underset{\uparrow}{p}}, \underset{\mathbb{R}^3}{\underset{\uparrow}{tp}})$$

It is one-to-one as $p \neq (0,0,0)$, and onto is clear.

Need: Show Φ and Φ^{-1} are C^∞.

$$\bar{F}^{-1} \circ \Phi \circ \tilde{F}(u_1,u_2,t) = \bar{F}^{-1} \circ \Phi\left(F(u_1,u_2),\ t\frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2}\right)$$

$$= \bar{F}^{-1} \circ \Phi\left(F(u_1,u_2),\ \underbrace{\frac{t\left(\frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2}\right)\cdot \hat{n}}{\cancel{|\hat{n}|^2}\ 1}}_{\text{as } \frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2} \parallel \hat{n}.}\ \hat{n}\right)$$

scalar

$$= \bar{F}^{-1}\left(F(u_1,u_2),\ \left(t\left(\frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2}\right)\cdot\hat{n}\right)F(u_1,u_2)\right)$$

$$= \Big(\underbrace{u_1, u_2}_{C^\infty},\ \underbrace{t\left(\frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2}\right)\cdot \hat{n}}_{C^\infty \text{ as } F \text{ and } \hat{n} \text{ are } C^\infty.}\Big)$$

$\therefore\ \Phi$ is C^∞.

$$\tilde{F}^{-1} \circ \Phi^{-1} \circ \bar{F}(u_1,u_2,t) = \tilde{F}^{-1} \circ \Phi^{-1}\left(F(u_1,u_2),\ tF(u_1,u_2)\right)$$

$$= \tilde{F}^{-1}\left(F(u_1,u_2),\ t\,\hat{n}(F(u_1,u_2))\right)$$

Scalar

$$= \tilde{F}^{-1}\left(F(u_1,u_2),\ t\,\frac{\hat{n}\cdot\left(\frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2}\right)}{\left|\frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2}\right|^2}\left(\frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2}\right)\right)$$

$$= \Big(u_1, u_2,\ \underbrace{t\,\frac{\hat{n}\cdot\left(\frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2}\right)}{\left|\frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2}\right|^2}}_{C^\infty \text{ as } \frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2} \neq 0.}\Big)$$

$\therefore\ \Phi^{-1}$ is C^∞.

Φ is a diffeomorphism. ∎

PROBLEM 3 HW3 Q6

(a) $\Sigma \xrightarrow{\iota_\Sigma} M\times N \xrightarrow{\pi_M} M$

$\underbrace{\iota_\Sigma^* \pi_M^* \Omega}_{} \xleftarrow{\iota_\Sigma^*} \pi_M^*\Omega \xleftarrow{\pi_M^*} \Omega$

↖ a differential form on Σ.

(b) First show $\exists$ such Φ, then check it is C^∞.

By bijectivity of $\pi_M \circ \iota_\Sigma : \Sigma \to M$, given any $p \in M$

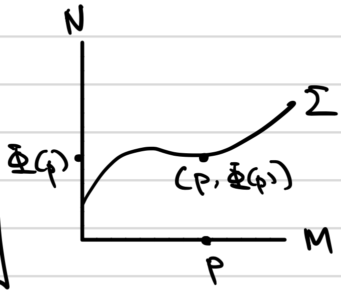
$\exists!\ \underbrace{(x(p), y(p))}_{\text{depending on } p} \in \Sigma \subset M\times N$ s.t. $\pi_M \circ \iota_\Sigma (x(p), y(p)) = p$

$\Rightarrow x(p) = p$.

Define $\Phi(p) := y(p) \quad \forall p \in M$.

This shows $\{(p, \Phi(p)) : p \in M\} \subset \Sigma$. — (*)

[Φ is well-defined as for every $p \in M$ There is one and only one point $(p, y(p)) \in \Sigma$ with p as the M-coordinate.]



Conversely, $\forall (p,q) \in \Sigma$, one can show $q = \Phi(p)$ by:

$p = \pi_M \circ \iota_\Sigma (p,q) = \pi_M \circ \iota_\Sigma (p, \Phi(p))$ (by definition of π_M)

$\Rightarrow (p,q) = (p, \Phi(p))$ (injectivity of $\pi_M \circ \iota_\Sigma$)

$\Rightarrow q = \Phi(p)$.

$\therefore$ Any point $(p,q) \in \Sigma$ is of the form $(p, \Phi(p))$

$\Rightarrow \Sigma \subset \{(p, \Phi(p)) : p \in M\}$ —(**)

(*) + (**) $\Rightarrow \Sigma = \{(p, \Phi(p)) : p \in M\}$.

Next we need to show Φ is C^∞.

Parametrize M by local coordinates $F(u_1, \dots, u_n) : \mathcal{U} \to M$

N by - - - - $G(v_1, \dots, v_n)$ - - -

$M\times N$ by — — — $(F\times G)(u_1, \dots, u_n, v_1, \dots, v_n)$

Σ by — — - . $H(w_1, \dots, w_n)$

Ω is a non-vanishing n-form on M

$\Rightarrow$ locally $\Omega = f\, du^1 \wedge \cdots \wedge du^n$ where $f(p) \neq 0$ $\forall p$ in the local coordinate chart.

$$\Rightarrow \pi_M^* \Omega = (f \circ \pi_M)(\pi_M^* du^1) \wedge \cdots \wedge (\pi_M^* du^n)$$

$$= (f \circ \pi_M)\, \underbrace{du^1 \wedge \cdots \wedge du^n}_{}$$

n-form on $M \times N$

(see below)

locally π_M maps

$(u_1, \ldots, u_n, v_1, \ldots, v_n)$

$\mapsto (u_1, \ldots, u_n)$

$$\Rightarrow \pi_M^* du^i\left(\frac{\partial}{\partial u_j}\right) = du^i\left(\pi_{M*} \frac{\partial}{\partial u_j}\right)$$

$$= du^i\left(\frac{\partial}{\partial u_j}\right)$$

$$= \delta_{ij}$$

$$(\pi_M^* du^i)\left(\frac{\partial}{\partial v_j}\right)$$

$$= du^i\left(\pi_{M*} \frac{\partial}{\partial v_j}\right)$$

$$= du^i(0) = 0.$$

$$\therefore \pi_M^* du^i = \sum_j \delta_{ij}\, du^j$$

$$= du^i$$

$\Rightarrow \iota_\Sigma^* \pi_M^* \Omega$

$$= \iota_\Sigma^* \left(f \circ \pi_M\, du^1 \wedge \cdots \wedge du^n\right)$$

$$= (f \circ \pi_M \circ \iota_\Sigma)(\iota_\Sigma^* du^1 \wedge \cdots \wedge \iota_\Sigma^* du^n)$$

$$= (f \circ \pi_M \circ \iota_\Sigma)\left(\sum_{i_1} \frac{\partial u_1}{\partial w_{i_1}} dw^{i_1}\right) \wedge \cdots \wedge \left(\sum_{i_n} \frac{\partial u_n}{\partial w_{i_n}} dw^{i_n}\right)$$

$\uparrow$ by $(\dagger)$ below.

$$= (f \circ \pi_M \circ \iota_\Sigma) \sum_{\substack{i_1, \ldots, i_n \\ \text{distinct}}} \frac{\partial u_1}{\partial w_{i_1}} \cdots \frac{\partial u_n}{\partial w_{i_n}}\, dw^{i_1} \wedge \cdots \wedge dw^{i_n}$$

$$= (f \circ \pi_M \circ \iota_\Sigma) \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \frac{\partial u_1}{\partial w_{\sigma(1)}} \cdots \frac{\partial u_n}{\partial w_{\sigma(n)}}\, dw^1 \wedge \cdots \wedge dw^n$$

$$= (f \circ \pi_M \circ \iota_\Sigma) \det \frac{\partial(u_1, \ldots, u_n)}{\partial(w_1, \ldots, w_n)}\, dw^1 \wedge \cdots \wedge dw^n \quad \longleftarrow \star$$

$(F \times G)^{-1} \circ \iota_\Sigma \circ H\,(w_1, \ldots, w_n)$

$= (u_1, \ldots, u_n, v_1, \ldots, v_n)$

$$\begin{array}{ccc} \Sigma & \xrightarrow{\iota_\Sigma} & M \times N \\ H \uparrow & & \uparrow F \times G \\ (w_1, \ldots, w_n) & & (u_1, \ldots, u_n, v_1, \ldots, v_n) \end{array}$$

$$\Rightarrow (\iota_\Sigma)_* \left(\frac{\partial}{\partial w_i}\right)$$

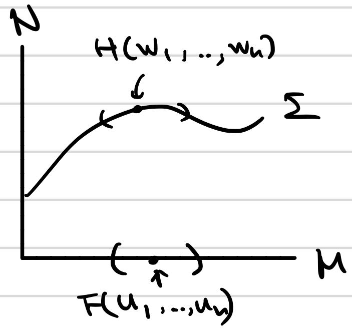
$$= \sum_j \left(\frac{\partial u_j}{\partial w_i} \frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial w_i} \frac{\partial}{\partial v_j}\right)$$

$$\Rightarrow (\iota_\Sigma^* du^k)\left(\frac{\partial}{\partial w_i}\right) = du^k\left((\iota_\Sigma)_* \frac{\partial}{\partial w_i}\right) = \frac{\partial u_k}{\partial w_i}$$

$$\Rightarrow \iota_\Sigma^* du^k = \sum_i \frac{\partial u_k}{\partial w_i} dw^i \qquad (\dagger)$$

$\uparrow$ dual of $\frac{\partial}{\partial w_i}$

Given that $\Omega(p) \neq 0$, $\iota_\Sigma^* \pi_M^* \Omega(p) \neq 0$ $\forall p \in M$,
we have
locally $\det \dfrac{\partial(u_1, \dots, u_n)}{\partial(w_1, \dots, w_n)} \neq 0$ on $F(\mathcal{U}) \subset M$.



Inverse function theorem
$\Rightarrow$ $(w_1, \dots, w_n)$ is locally a C^∞ function
of $(u_1, \dots, u_n)$

say $(w_1, \dots, w_n) = \underset{C^\infty}{\psi}(u_1, \dots, u_n)$.

Every point $H(w_1, \dots, w_n) \in \Sigma$ is in the form of

$$H(w_1, \dots, w_n) = \Big(\underbrace{F(u_1, \dots, u_n)}_{\text{in } M}, \underbrace{\Phi(F(u_1, \dots, u_n))}_{\text{in } N}\Big)$$

$$\Rightarrow \Phi(F(u_1, \dots, u_n)) = \pi_N \circ H(w_1, \dots, w_n)$$

$$\Rightarrow G^{-1} \circ \Phi \circ F(u_1, \dots, u_n) = \underbrace{G^{-1} \circ \pi_N \circ H}_{C^\infty \text{ as } \pi_N \text{ is } C^\infty} \circ \underbrace{\psi(u_1, \dots, u_n)}_{C^\infty}.$$

Hence Φ is C^∞ locally.

Discussion:

(i) Ω needs to be an n-form
as the fundamental result:

$$\iota_\Sigma^* \pi_M^* \Omega = (f \circ \pi_M \circ \iota_\Sigma) \det \frac{\partial(u_1, \dots, u_n)}{\partial(w_1, \dots, w_n)} dw^1 \wedge \dots \wedge dw^n \quad \text{— } \maltese$$

would not hold if Ω were not an n-form
(more terms would appear if Ω is k-form, $k < n$)

(ii) If Ω is somewhere 0, then $\iota_\Sigma^* \pi_M^* \Omega$ is somewhere 0 too. We need $\iota_\Sigma^* \pi_M^* \Omega$ is nowhere zero to show that $\det \dfrac{\partial(u_1, \dots, u_n)}{\partial(w_1, \dots, w_n)} \neq 0$ so that inverse function theorem applies.

(iii) Well-definedness of Φ follows from injectivity of $\iota_\Sigma \circ \pi_M$.

(c) (i) $M\times N \xrightarrow{\pi_M} M$ $\qquad$ $M\times N \xrightarrow{\pi_N} N$

$\pi_M^*\omega_M \xleftarrow{\pi_M^*} \omega_M$ $\qquad$ $\pi_N^*\omega_N \xleftarrow{\pi_N^*} \omega_N$

Both $\pi_M^*\omega_M$ and $\pi_N^*\omega_N$ are forms on $M\times N$

$\Rightarrow$ so does η.

(ii) From (b), $\Sigma = \{(p, \Phi(p)) : p\in M\}$ where $\Phi: M\to N$ is C^∞.

Parametrize M by $F(u_1,\dots,u_n)$, N by $G(v_1,\dots,v_n)$

then Σ can now be parametrized by

$$\tilde F(u_1,\dots,u_n) = (\underbrace{F(u_1,\dots,u_n)}_{M}, \underbrace{\Phi(F(u_1,\dots,u_n))}_{N})$$

$$T\Sigma = \operatorname{span}\left\{\frac{\partial}{\partial u_i}\right\}_{i=1}^n$$

$$(F\times G)^{-1}\circ \iota_\Sigma\circ \tilde F(u_1,\dots,u_n)$$
$$= (F\times G)^{-1}(F(u_1,\dots,u_n), \Phi(F(u_1,\dots,u_n)))$$
$$= (u_1,\dots,u_n, G^{-1}\circ\Phi\circ F(u_1,\dots,u_n))$$

$$\Rightarrow (\iota_\Sigma)_*\left(\frac{\partial}{\partial u_i}\right) = \underbrace{\frac{\partial}{\partial u_i}}_{\text{in } TM} + \underbrace{\frac{\partial \Phi}{\partial u_i}}_{\text{in } TN} = \frac{\partial}{\partial u_i} + \Phi_*\left(\frac{\partial}{\partial u_i}\right)$$

Special case for 1-forms. General case for k-forms is the same.

$$\Rightarrow (\iota_\Sigma^*\eta)\left(\frac{\partial}{\partial u_i}\right) = \eta\left((\iota_\Sigma)_*\frac{\partial}{\partial u_i}\right) = \eta\left(\frac{\partial}{\partial u_i} + \Phi_*\frac{\partial}{\partial u_i}\right)$$
$$= (\pi_M^*\omega_M - \pi_N^*\omega_N)\left(\frac{\partial}{\partial u_i} + \Phi_*\frac{\partial}{\partial u_i}\right)$$
$$= \omega_M\left(\pi_{M*}\left(\underbrace{\frac{\partial}{\partial u_i}}_{\text{in } TM} + \underbrace{\Phi_*\frac{\partial}{\partial u_i}}_{\text{in } TN}\right)\right) - \omega_N\left(\pi_{N*}\left(\underbrace{\frac{\partial}{\partial u_i}}_{\text{in } TM} + \underbrace{\Phi_*\frac{\partial}{\partial u_i}}_{\text{in } TN}\right)\right)$$
$$= \omega_M\left(\frac{\partial}{\partial u_i}\right) - \omega_N\left(\Phi_*\frac{\partial}{\partial u_i}\right)$$
$$= (\omega_M - \Phi^*\omega_N)\left(\frac{\partial}{\partial u_i}\right)$$

$\therefore\ \iota_\Sigma^*\eta \equiv 0 \iff \omega_M \equiv \Phi^*\omega_N$ ∎

DONE ☺