Exterior derivative

$f \in C^\infty(M,\mathbb{R}) =: \Lambda^0 T^*M$, $\{u_1,\dots,u_n\}$ local coordinates, $\{v_1,\dots,v_n\}$

$df := \sum_{i=1}^{n} \frac{\partial f}{\partial u_i} du^i \in \Lambda^1 T^*M$.

$\frac{\partial f}{\partial u_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial v_j}\frac{\partial v_j}{\partial u_i}$ $\qquad du^i = \sum_{k=1}^{n} \frac{\partial u^i}{\partial v_k} dv^k$

$$\sum_{i=1}^{n} \frac{\partial f}{\partial u_i} du^i = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial f}{\partial v_j}\frac{\partial v_j}{\partial u_i} \sum_{k=1}^{n}\frac{\partial u^i}{\partial v_k} dv^k = \sum_{\substack{j=1\\k=1}}^{n} \frac{\partial f}{\partial v_j} \underbrace{\frac{\partial v_j}{\partial v_k}}_{\delta_{kj}} dv^k = \sum_{j=1}^{n}\frac{\partial f}{\partial v_j} dv^j.$$

$\cancel{\nabla f \text{ "=" } \sum_j \frac{\partial f}{\partial u_j}\frac{\partial}{\partial u_j}}$

$\alpha = \sum_i \alpha_i du^i \in \Lambda^1 T^*M$ $\qquad \alpha_i : U \subset M \to \mathbb{R}$.

$d\alpha := \sum_i \underbrace{d\alpha_i}_{\text{scalar function}} \wedge du^i = \sum_{i,j} \underbrace{\frac{\partial \alpha_i}{\partial u_j} du^j}_{d\alpha_i} \wedge du^i$

$\mathbb{R}^3$ (x,y,z). $f : \mathbb{R}^3 \to \mathbb{R}$

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$

$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$\alpha = \alpha_x dx + \alpha_y dy + \alpha_z dz$

$d\alpha = d\alpha_x \wedge dx + d\alpha_y \wedge dy + d\alpha_z \wedge dz$

$\underbrace{d\alpha_x}_{\text{scalar function}} = \frac{\partial \alpha_x}{\partial x}dx + \frac{\partial \alpha_x}{\partial y}dy + \frac{\partial \alpha_x}{\partial z}dz$.

$\Rightarrow d\alpha_x \wedge dx = \underset{\frac{\partial \alpha_x}{\partial x}dx\wedge dx}{0} + \frac{\partial \alpha_x}{\partial y}dy \wedge dx + \frac{\partial \alpha_x}{\partial z}dz \wedge dx$

$d\alpha_y \wedge dy = \left(\frac{\partial \alpha_y}{\partial x}dx + \frac{\partial \alpha_y}{\partial y}\cancel{dy} + \frac{\partial \alpha_y}{\partial z}dz\right)\wedge dy = \frac{\partial \alpha_y}{\partial x}dx\wedge dy + \frac{\partial \alpha_y}{\partial z}dz\wedge dy$

$d\alpha_z \wedge dz = \frac{\partial \alpha_z}{\partial x}dx\wedge dz + \frac{\partial \alpha_z}{\partial y}dy\wedge dz$.

$d\alpha = \left(\frac{\partial \alpha_y}{\partial x} - \frac{\partial \alpha_x}{\partial y}\right)dx\wedge dy + \left(\frac{\partial \alpha_x}{\partial z} - \frac{\partial \alpha_z}{\partial x}\right)dz\wedge dx + \left(\frac{\partial \alpha_z}{\partial y} - \frac{\partial \alpha_y}{\partial z}\right)dy \wedge dz$

$= \left(\frac{\partial \alpha_y}{\partial x} - \frac{\partial \alpha_x}{\partial y}\right)\hat{k} + \left(\frac{\partial \alpha_x}{\partial z} - \frac{\partial \alpha_z}{\partial x}\right)\hat{j} + \left(\frac{\partial \alpha_z}{\partial y} - \frac{\partial \alpha_y}{\partial z}\right)\hat{i}$

$V = \alpha_x \hat{i} + \alpha_y \hat{j} + \alpha_z \hat{k}$

$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \alpha_x & \alpha_y & \alpha_z \end{vmatrix}$

$\omega = \sum \underbrace{\omega_{i_1\cdots i_k}}_{\text{scalar function.}} du^{i_1}\wedge\cdots\wedge du^{i_k}$

$d\omega := \sum d\omega_{i_1\cdots i_k} \wedge du^{i_1}\wedge\cdots\wedge du^{i_k}$ $\qquad \in T^*M$. $\qquad d : C^\infty(U,\mathbb{R}) \to \Lambda^1 T^*U$.

$d : \Lambda^k T^*M \to \Lambda^{k+1}T^*M$

$\mathbb{R}^3, (x,y,z)$

$\beta = P\, dy\wedge dz + Q\, dz \wedge dx + R\, dx\wedge dy$

$\vec{G} = P\,\hat{i} + Q\,\hat{j} + R\,\hat{k}$

$d\beta = dP\wedge dy\wedge dz + dQ\wedge dz\wedge dx + dR\wedge dx\wedge dy$

$= \left(\frac{\partial P}{\partial x}dx + \cancel{\frac{\partial P}{\partial y}dy} + \cancel{\frac{\partial P}{\partial z}dz}\right)\wedge dy\wedge dz + \frac{\partial Q}{\partial y}dy\wedge dz\wedge dx + \frac{\partial R}{\partial z}dz\wedge dx\wedge dy$.

$= \frac{\partial P}{\partial x}dx\wedge dy\wedge dz + \frac{\partial Q}{\partial y}dx\wedge dy\wedge dz + \frac{\partial R}{\partial z}dx\wedge dy\wedge dz$

$= \underbrace{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)}_{\nabla\cdot\vec{G} = \operatorname{div}(\vec{G})} dx\wedge dy\wedge dz$

$\Lambda^0 T^*M \xrightarrow[d]{d_0} \Lambda^1 T^*M \xrightarrow[d]{d_1} \Lambda^2 T^*M \xrightarrow[d]{d_2} \Lambda^3 T^*M \xrightarrow[d]{d_3} \cdots \xrightarrow[d]{d_{n-1}} \Lambda^n T^*M$

$\boxed{d^2 = 0}$ $\leftarrow$ $d_{k+1} \circ d_k = 0 \quad \forall k$.

Prop. 3.43 $\qquad d\circ d = 0$

$\omega = \omega_{i_1\cdots i_k} du^{i_1}\wedge\cdots\wedge du^{i_k}$

$d\omega = d\omega_{i_1\cdots i_k} \wedge du^{i_1}\wedge\cdots\wedge du^{i_k}$

$= \sum_j \frac{\partial \omega_{i_1\cdots i_k}}{\partial u_j} du^j\wedge du^{i_1}\wedge\cdots\wedge du^{i_k}$

$d^2\omega = d(d\omega)$

$= \sum_j d\left(\frac{\partial \omega_{i_1\cdots i_k}}{\partial u_j}\right)\wedge du^j \wedge du^{i_1}\wedge\cdots\wedge du^{i_k}$

$= \sum_{j,k} \underbrace{\frac{\partial^2 \omega_{i_1\cdots i_k}}{\partial u_k \partial u_j}}_{k\leftrightarrow j} \underbrace{du^k\wedge du^j}_{\text{anti-sym.}} \wedge du^{i_1}\wedge\cdots\wedge du^{i_k}$

$= \left(\sum_{j<k} \frac{\partial^2 \omega_{i_1\cdots i_k}}{\partial u_k \partial u_j} du^k\wedge du^j + \sum_{k<j} \frac{\partial^2 \omega_{i_1\cdots i_k}}{\partial u_k \partial u_j} du^k\wedge du^j\right)\wedge du^{i_1}\wedge\cdots\wedge du^{i_k}$

$= \left(\sum_{j<k} \frac{\partial^2 \omega_{i_1\cdots i_k}}{\partial u_k \partial u_j} du^k\wedge du^j + \sum_{j<k} \frac{\partial^2 \omega_{i_1\cdots i_k}}{\partial u_j \partial u_k} \underbrace{du^j\wedge du^k}_{= -du^k\wedge du^j}\right)\wedge du^{i_1}\wedge\cdots\wedge du^{i_k}$

$= 0$.

$\mathbb{R}^3$ $\quad$ 4033 $\quad$ 2023

$f \leftrightarrow f$

$df \leftrightarrow \nabla f$

$\alpha \in \Lambda^1 T^*M \leftrightarrow \vec{V}$

$d\alpha \leftrightarrow \nabla\times\vec{V}$.

$d(d\alpha) = 0 \leftrightarrow \nabla\cdot(\nabla\times\vec{V}) = 0$.

$d(df) = 0 \leftrightarrow \nabla\times(\nabla f) = 0$.

$\Lambda^{k-1}T^*M \xrightarrow{d_{k-1}} \Lambda^k T^*M \xrightarrow{d_k} \Lambda^{k+1}T^*M$ $\qquad d\circ d = 0$.

$d\alpha \in \operatorname{Im}(d_{k-1}) = \{\underbrace{d\alpha}_{\Lambda^k} : \alpha \in \Lambda^{k-1}\}$.

$d(d\alpha) = 0$

$d\alpha \in \operatorname{Ker}(d_k) = \{\beta \in \Lambda^k : d\beta = 0\}$

$\operatorname{Im}(d) \subset \operatorname{Ker}(d)$.

$H^k_{dR}(M) := \operatorname{Ker}(d) / \operatorname{Im}(d)$.

kth de Rham cohomology group of M.

topological invariant (!).

$\omega \in \Lambda^k$ is exact $\overset{\text{def}}{\Leftrightarrow}$ $\omega = d\eta$ $\quad \exists \eta \in \Lambda^{k-1}$

$\omega \in \Lambda^{k+1}$ is closed $\overset{\text{def}}{\Leftrightarrow}$ $d\omega = 0$

$\mathbb{R}^3$ $\left\{\begin{array}{l} \alpha \quad \text{1-form exact} \Leftrightarrow \alpha = df \\ \updownarrow \qquad\qquad\qquad\qquad \updownarrow \\ V \quad \text{vector field} \qquad V = \nabla f \end{array}\right.$

$\left\{\begin{array}{l} \alpha \quad \text{1-form closed} \Leftrightarrow d\alpha = 0 \\ \updownarrow \qquad\qquad\qquad\qquad \updownarrow \\ V \quad \text{vector field} \qquad \nabla\times\vec{V} = \vec{0}. \end{array}\right.$

$H^1_{dR}(M) = \operatorname{Ker}(d) / \operatorname{Im}(d) = \frac{\{\text{closed 1-forms}\}}{\{\text{exact 1-forms}\}}$.

- $d(\underset{\text{const}}{c}\omega + \eta) = c\,d\omega + d\eta$ $\qquad \forall \omega,\eta \in \Lambda^k T^*M$.
- $d(f\omega) = df\wedge\omega + f\,d\omega$.

  $f\cdot \omega_{i_1\cdots i_k} du^{i_1}\wedge\cdots\wedge du^{i_k}$
- $d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^k \omega\wedge d\eta$ $\quad \Big| \quad \forall \omega \in \Lambda^k T^*M,\ \eta \in \Lambda^r T^*M$.

Pull-back:

$\Phi^*(T_1\otimes\cdots\otimes T_n)$ $\qquad T_i \in T^*M$. $\quad \Phi : N \to M$.

$:= \Phi^*T_1 \otimes \Phi^*T_2 \otimes\cdots\otimes \Phi^*T_n$.

equivalently:

$(\Phi^*(T_1\otimes\cdots\otimes T_n))(\underbrace{Y_1,\dots,Y_n}_{TN}) = (T_1\otimes\cdots\otimes T_n)(\underbrace{\Phi_*Y_1,\dots,\Phi_*Y_n}_{TM})$

$(\Phi^*T_1 \otimes \Phi^*T_2 \otimes\cdots\otimes \Phi^*T_n)(Y_1,\dots,Y_n)$

$= (\Phi^*T_1)(Y_1)\cdots(\Phi^*T_n)(Y_n) = T_1(\Phi_*Y_1)\cdots T_n(\Phi_*Y_n)$

$\Phi^*\alpha = \Phi^*\left(\sum_i \alpha_i du^i\right)$

$= \sum_i \alpha_i \Phi^*(du^i)$

$\Phi : N \to M$ (v_α, u_i) $\qquad = \sum_i \alpha_i\left(\sum_\alpha \frac{\partial u^i}{\partial v_\alpha} dv^\alpha\right)$

$\iota : N \to M$ (submfd.) $\qquad N \subset M$.

$p \mapsto p$.

$\iota^*\alpha$ $\qquad \alpha \in T^*M$. u_i

$\iota^*\alpha \in T^*N$. v_α.

$\alpha = \sum_i \alpha_i du^i$

$\iota^*\alpha = \sum_\alpha \beta_\alpha dv^\alpha$

e.g. $\Sigma^2 \subset \mathbb{R}^3$ regular surface

$(u,v) \xrightarrow{F} \Sigma$, $\uparrow \text{id}$ (x,y,z).

$\iota^*dx = A\,du + B\,dv$

$(\iota^*dx)\left(\frac{\partial}{\partial u}\right) = (dx)\left(\iota_*\frac{\partial}{\partial u}\right)$ $\qquad \vec{F}(u,v) = (x(u,v), y(u,v), z(u,v))$

$= (dx)\left(\frac{\partial \vec{F}}{\partial u}\right)$

$= (dx)\left(\frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \cdots\right)$

$= \frac{\partial x}{\partial u}$.

$\Rightarrow \iota^*dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$.

$\iota^*(dx\otimes dx) = (\iota^*dx)\otimes(\iota^*dx)$

$= \left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right)\otimes\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right)$

$= \frac{\partial x}{\partial u}\frac{\partial x}{\partial u}du\otimes du + \frac{\partial x}{\partial v}\frac{\partial x}{\partial u}dv\otimes du + \frac{\partial x}{\partial u}\frac{\partial x}{\partial v}du\otimes dv + \frac{\partial x}{\partial v}\frac{\partial x}{\partial v}dv\otimes dv$.

$\delta = dx\otimes dx + dy\otimes dy + dz\otimes dz$. (dot product on $\mathbb{R}^3$)

$\iota^*\delta = \frac{\partial x}{\partial u}\frac{\partial x}{\partial u}du\otimes du + \frac{\partial x}{\partial v}\frac{\partial x}{\partial u}dv\otimes du$
$+ \frac{\partial x}{\partial u}\frac{\partial x}{\partial v}du\otimes dv + \frac{\partial x}{\partial v}\frac{\partial x}{\partial v}dv\otimes dv$
$+ \frac{\partial y}{\partial u}\frac{\partial y}{\partial u}du\otimes du + \frac{\partial y}{\partial v}\frac{\partial y}{\partial u}dv\otimes du$
$+ \frac{\partial y}{\partial u}\frac{\partial y}{\partial v}du\otimes dv + \frac{\partial y}{\partial v}\frac{\partial y}{\partial v}dv\otimes dv$
$+ \frac{\partial z}{\partial u}\frac{\partial z}{\partial u}du\otimes du + \frac{\partial z}{\partial v}\frac{\partial z}{\partial u}dv\otimes du$
$+ \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}du\otimes dv + \frac{\partial z}{\partial v}\frac{\partial z}{\partial v}dv\otimes dv$.

$= \left(\frac{\partial \vec{F}}{\partial u}\cdot\frac{\partial \vec{F}}{\partial u}\right)du\otimes du + \left(\frac{\partial \vec{F}}{\partial u}\cdot\frac{\partial \vec{F}}{\partial v}\right)du\otimes dv + \left(\frac{\partial \vec{F}}{\partial v}\cdot\frac{\partial \vec{F}}{\partial u}\right)dv\otimes du + \left(\frac{\partial \vec{F}}{\partial v}\cdot\frac{\partial \vec{F}}{\partial v}\right)dv\otimes dv$.

$g = \iota^*\delta$ first fundamental form of Σ.

Elie Cartan

$\Phi^*(T_1\wedge\cdots\wedge T_k)$ $\qquad T_i \in \Lambda^k T^*M$

$= (\Phi^*T_1)\wedge\cdots\wedge(\Phi^*T_k)$ $\qquad \Phi : N \to M$

$(u,v) \xrightarrow{F} \Sigma \xhookrightarrow{\iota} \mathbb{R}^3$ Regular surface.

$\iota^*(dx\wedge dy) = (\iota^*dx)\wedge(\iota^*dy)$ $\qquad \iota^*dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$.

$= \left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right)\wedge\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)$

$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}du\wedge dv + \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}dv\wedge du$

$= \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)du\wedge dv$.

$= \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix} du\wedge dv$

$= \det\frac{\partial(x,y)}{\partial(u,v)}\, du\wedge dv$

$\vec{V} = P\hat{i} + Q\hat{j} + R\hat{k}$

$\iota^*(P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy)$

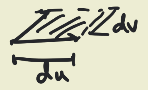
$= \left(P\det\frac{\partial(y,z)}{\partial(u,v)} + Q\det\frac{\partial(z,x)}{\partial(u,v)} + R\det\frac{\partial(x,y)}{\partial(u,v)}\right)du\wedge dv$.

$= (P\hat{i} + Q\hat{j} + R\hat{k})\cdot\left(\det\frac{\partial(y,z)}{\partial(u,v)}\hat{i} + \det\frac{\partial(z,x)}{\partial(u,v)}\hat{j} + \det\frac{\partial(x,y)}{\partial(u,v)}\hat{k}\right)du\wedge dv$

$= \vec{V}\cdot\left(\frac{\partial\vec{F}}{\partial u}\times\frac{\partial\vec{F}}{\partial v}\right)du\wedge dv$

$\frac{\partial\vec{F}}{\partial u}\times\frac{\partial\vec{F}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$ $\qquad \vec{F}(u,v) = (x(u,v), y(u,v), z(u,v))$

$= \vec{V}\cdot\frac{\frac{\partial\vec{F}}{\partial u}\times\frac{\partial\vec{F}}{\partial v}}{\left|\frac{\partial\vec{F}}{\partial u}\times\frac{\partial\vec{F}}{\partial v}\right|}\cdot\underbrace{\left|\frac{\partial\vec{F}}{\partial u}\times\frac{\partial\vec{F}}{\partial v}\right| du\wedge dv}$.



$= \vec{V}\cdot\hat{\nu}_{\text{unit}}\, dS$.

$\Phi : M \to N$ $\qquad M : (u_1,\dots,u_m)$, $\quad N : (v_1,\dots,v_n)$

$\left[\Phi^* d_N\omega = d_M \Phi^* \underset{\Lambda^k T^*N}{\omega}\right]$ $\qquad \Phi(u_1,\dots,u_m) = (v_1,\dots,v_n)$

$\Phi^* d_N f = \Phi^*\left(\sum_j \frac{\partial f}{\partial v_j}dv^j\right)$ $\qquad \Phi^*\left(\frac{\partial f}{\partial v_j}\right)$

$= \sum_j\left(\frac{\partial f}{\partial v_j}\circ\Phi\right)\Phi^*dv^j = \sum_j\left(\frac{\partial f}{\partial v_j}\circ\Phi\right)\sum_i\frac{\partial v^j}{\partial u_i}du^i$

$\Phi^*f := f\circ\Phi$. $\qquad = \sum_{j,i}\left(\frac{\partial f}{\partial v_j}\circ\Phi\right)\frac{\partial v^j}{\partial u_i}du^i$.

$d_M(\Phi^*f) = d(f\circ\Phi) = \sum_i\frac{\partial(f\circ\Phi)}{\partial u_i}du^i$ $\qquad f\circ\Phi$, v_j, u_j

$= \sum_{i,j}\left(\frac{\partial f}{\partial v_j}\circ\Phi\right)\frac{\partial v^j}{\partial u_i}du^i$

$\Rightarrow \Phi^*df = d(\Phi^*f)$.

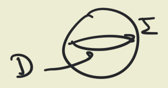
$\vec{V} = P\hat{i} + Q\hat{j} + R\hat{k}$

$\updownarrow$

$\beta = P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy$.

$d\beta = (\operatorname{div}\vec{V})\,dx\wedge dy\wedge dz$.

Div. Thm: $\qquad \underbrace{\oint_\Sigma \vec{V}\cdot\hat{\nu}\,dS}_{\|} = \iiint_D (\operatorname{div}\vec{V})\,dx\,dy\,dz$. $\|$



$$\boxed{\oint_{\Sigma = \partial D} \iota^*\beta = \iiint_D d\beta.}$$

$I(M,\beta) := \int_M \beta$ $\qquad I(\partial D, \iota^*\beta) = I(D, d\beta)$

Stokes': $\qquad \vec{V} = \langle P,Q,R\rangle$.

$\underbrace{\oint_{C=\partial S} P\,dx + Q\,dy + R\,dz} = \iint_S (\nabla\times\vec{V})\cdot\hat{\nu}\,dS$

$\alpha = P\,dx + Q\,dy + R\,dz$

$\iota_C : C \hookrightarrow \mathbb{R}^3$

$\iota_S : S \hookrightarrow \mathbb{R}^3$

$\iota_{C,S} : C \hookrightarrow S$

$\oint_C \iota_C^*\alpha = \int_S \iota_S^*(d\alpha) = \int_S d(\iota_S^*\alpha)$

$$\boxed{\oint_{C=\partial S}\alpha = \iint_S d\alpha.}$$