MATH 4033 • Spring 2019 • Calculus on Manifolds
Problem Set #3 • Tensors and Differential Forms • Due Date: 31/03/2019, 11:59PM

1. (10 points) Let M^n be a C^∞ n-dimensional manifold. For each $p \in M$, we define the $(1,1)$-tensor space at p by:

$$T_p^{1,1}M := T_p^*M \otimes T_pM = \text{span}\left\{ du^i\big|_p \otimes \frac{\partial}{\partial u_j}(p) \right\}_{i,j=1}^n,$$

and similar to tangent and cotangent bundles, we define the $(1,1)$-tensor bundle of M by:

$$T^{1,1}M := \bigcup_{p \in M} \{p\} \times T_p^{1,1}M.$$

   (a) Show that $T^{1,1}M$ is a C^∞ manifold. What is $\dim T^{1,1}M$?

   (b) Show that if $n = 1$, then $T^{1,1}M$ is diffeomorphic to $M \times \mathbb{R}$.

2. (15 points) Consider the maps $\Phi_i : \mathbb{R} \times (0, 2\pi) \to \mathbb{R}^3$, where $i = 1, 2$, defined as:

$$\Phi_1(u, v) = (\cosh u \cos v, \cosh u \sin v, v),$$
$$\Phi_2(r, \theta) = (r \cos \theta, r \sin \theta, \theta).$$

   (a) Denote $\delta = dx \otimes dx + dy \otimes dy + dz \otimes dz$, where x, y, z are the usual Cartesian coordinates on $\mathbb{R}^3$. Compute $\Phi_i^*\delta$ for each i.

   (b) Show that there exists a C^∞ map $\psi : \mathbb{R} \times (0, 2\pi) \to \mathbb{R} \times (0, 2\pi)$ such that

$$(\Phi_2 \circ \psi)^*\delta = \Phi_1^*\delta.$$

3. (20 points) Consider the following 2-tensor and 3-form defined on $M := \underbrace{(1, \infty)}_{s} \times \underbrace{\mathbb{S}^2}_{\phi,\theta}$:

$$g := \frac{s}{s^3 + s - 2}\, ds \otimes ds + s^2(d\phi \otimes d\phi + \sin^2\phi \, d\theta \otimes d\theta)$$
$$\Omega := \left(\frac{s^5 \sin^2\phi}{s^3 + s - 2} \right)^{\frac{1}{2}} ds \wedge d\phi \wedge d\theta = \sqrt{\det[g]}\, ds \wedge d\phi \wedge d\theta$$

Here $\mathbb{S}^2$ is the unit sphere, and (ϕ, θ) are the standard spherical coordinates using math convention: $\phi \in (0, \pi)$ and $\theta \in (0, 2\pi)$.

   (a) Let $X = \frac{\partial}{\partial\theta}$ and $Y = \frac{\partial}{\partial\phi}$. Compute all of the following:

$$\mathcal{L}_X g, \ \ \mathcal{L}_Y g, \ \ i_X\Omega, \ \ i_Y\Omega, \ \ \mathcal{L}_X\Omega, \ \ \mathcal{L}_Y\Omega.$$

   (b) Show that there *exists* a local coordinate system (r, ϕ, θ) of M such that:

$$g = dr \otimes dr + f(r)^2(d\phi \otimes d\phi + \sin^2\phi \, d\theta \otimes d\theta)$$

   for some positive smooth function $f(r)$.

4. (15 points) (a) Suppose the following 1-form on $\mathbb{R}^3$ is closed:

$$\omega = pdx + qdy + rdz$$

where $p, q, r : \mathbb{R}^3 \to \mathbb{R}$ are homogeneous smooth functions of of degree m. Show that $\omega = df$ where $f = \dfrac{xp + yq + zr}{m+1}$, i.e. ω is exact.

(b) Suppose that the following 2-form on $\mathbb{R}^3$ is closed:

$$\Omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$$

where $P, Q, R : \mathbb{R}^3 \to \mathbb{R}$ are homogeneous smooth functions of degree m. Show that $\Omega = d\alpha$, where

$$\alpha = \frac{(zQ - yR)dx + (xR - zP)dy + (yP - xQ)dz}{m+2}.$$

[FYI: In fact all smooth closed 1-forms and 2-forms on $\mathbb{R}^3$ are exact, but the primitive forms are not as explicit as the above if p, q, r and P, Q, R are not homogeneous functions.

5. (15 points) Exercises 3.56 and 3.57 (about converting the four Maxwell's equations into two elegant equations using differential forms).

6. (25 points) Consider smooth manifolds M and N with the same dimension n. Suppose $\Omega \in \wedge^n T^*M$ is a C^∞ n-form on M such that $\Omega(p) \neq 0$ for any $p \in M$. Given an n-dimensional submanifold Σ of $M \times N$, we denote:

- $\iota_\Sigma : \Sigma \to M \times N$ to be the inclusion map
- $\pi_M : M \times N \to M$ to be the projection map $(p, q) \in M \times N \mapsto p \in M$.
- $\pi_N : M \times N \to N$ to be the projection map $(p, q) \in M \times N \mapsto q \in N$.

(a) Is $\iota_\Sigma^* \pi_M^* \Omega$ a differential form on M, N, $M \times N$, or Σ? Explain briefly your answer.

(b) Show that if $\iota_\Sigma^* \pi_M^* \Omega$ is nowhere zero and $\pi_M \circ \iota_\Sigma$ is bijective, then there exists a well-defined C^∞ map $\Phi : M \to N$ such that $\Sigma = \{(p, \Phi(p)) \in M \times N : p \in M\}$, i.e. Σ is the graph of Φ. [Hint: You may need the inverse/implicit function theorem.]

(c) Assume the condition given in (b) so that Σ is the graph of a C^∞ map $\Phi : M \to N$. Given a C^∞ k-form ω_M on M, and a C^∞ k-form ω_N on N, we define:

$$\eta := \pi_M^* \omega_M - \pi_N^* \omega_N.$$

i. Is η a differential form on M, N, $M \times N$, or Σ? Explain briefly your answer.

ii. Show that $\iota_\Sigma^* \eta \equiv 0$ if and only if $\omega_M \equiv \Phi^* \omega_N$.