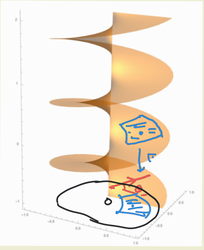


helicoid $= \Sigma_{hel.} \underset{\text{diffeo}}{\overset{\text{local}}{\cong}} \mathbb{R}^2 \setminus \{0\}$.

$F(r,\theta) : (0,\infty) \times \mathbb{R} \to \Sigma_{hel} \subset \mathbb{R}^3$

$= (r\cos\theta, r\sin\theta, \theta)$

$$\begin{array}{ccc} & \Sigma_{hel} \xrightarrow{\ \pi\ } \mathbb{R}^2\setminus\{0\} \nearrow id \\ F \nearrow & (x,y,z) \mapsto (x,y) \\ (r,\theta) & \end{array}$$

$id^{-1} \circ \pi \circ F(r,\theta) = (r\cos\theta, r\sin\theta)$.

$$D(id^{-1}\circ\pi\circ F) = \frac{\partial(r\cos\theta, r\sin\theta)}{\partial(r,\theta)} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$
$\frac{\partial}{\partial r}$ $\quad \frac{\partial}{\partial \theta}$

$\det D(id^{-1}\circ\pi\circ F) = r^2 \neq 0$.

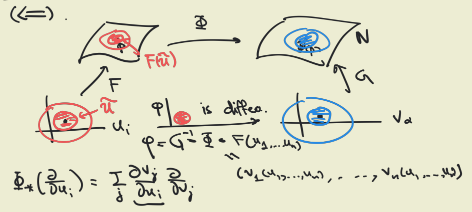
$\Rightarrow (id^{-1}\circ\pi\circ F)^{-1}$ is C^∞.
$= F^{-1}\circ\pi^{-1}\circ id$

Inverse Function Theorem for manifolds:

$\Phi : M \to N$ is a local diffeomorphism

$\Leftrightarrow$ $(\Phi_*)_p : T_pM \to T_{\Phi(p)}N$ is invertible for any $p\in M$.

Proof:
$(\Leftarrow)$.



$\varphi = G^{-1}\circ\Phi\circ F(u_1,\dots,u_n)$
$= (v_1(u_1,\dots,u_n), \dots, v_n(u_1,\dots,u_n))$

$\varphi|$ is diffeo.

$$\Phi_*\left(\frac{\partial}{\partial u_i}\right) = \sum_j \frac{\partial v_j}{\partial u_i}\frac{\partial}{\partial v_j}$$

$$[\Phi_*] = \frac{\partial(v_1,\dots,v_n)}{\partial(u_1,\dots,u_n)} = D(G^{-1}\circ\Phi\circ F).$$

Claim: $\Phi|_{F(\tilde{u})}$ is a diffeo.

$G^{-1}\circ\Phi|_{F(\tilde{u})}\circ F = G^{-1}\circ\Phi\circ F|_{\tilde{u}}$

$\Phi|_{F(\tilde{u})}^{-1} = F\circ\varphi^{-1}\circ G|^{-1}$

$F^{-1}\circ\Phi|_{F(\tilde{u})}^{-1}\circ G = \varphi^{-1}$.

Immersion and Submersion.

$\Phi : M \to N$.

Def: Φ is an immersion at $p\in M$
$\overset{\text{def}}{\Leftrightarrow}$ $(\Phi_*)_p$ is injective.

Φ is a submersion at $p\in M$
$\overset{\text{def}}{\Leftrightarrow}$ $(\Phi_*)_p$ is surjective.

e.g.
$\mathbb{RP}^1 \times \mathbb{RP}^2 \xrightarrow{\Phi} \mathbb{RP}^5$

$[x_0:x_1], [y_0:y_1:y_2] \mapsto [x_0y_0 : x_0y_1 : x_0y_2 : x_1y_0 : x_1y_1 : x_1y_2]$

$(F\times G)(u, v_1, v_2) = ([1:u],[1:v_1:v_2])$ $\qquad$ $H(w_1,\dots,w_5) = [1:w_1:\dots:w_5]$

$\Phi_*\left(\frac{\partial}{\partial u}\right)$, $\Phi_*\left(\frac{\partial}{\partial v_1}\right)$, $\Phi_*\left(\frac{\partial}{\partial v_2}\right)$

$H^{-1}\circ\Phi\circ(F\times G)(u,v_1,v_2) = (\ \dots \ )$

$$D(H^{-1}\circ\Phi\circ(F\times G)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ v_1 & u & 0 \\ v_2 & 0 & u \end{bmatrix}$$
$\frac{\partial(\ )}{\partial u}$ $\quad \frac{\partial(\ )}{\partial v_1}$ $\quad \frac{\partial(\ )}{\partial v_2}$

$$[T] = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ & * & \end{bmatrix}$$
linearly independent $\Rightarrow T$ is injective.

$$\begin{bmatrix} ? & 1 & 2 & 3 \\ ? & 4 & 5 & 6 \\ ? & 7 & 8 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 8 \end{bmatrix}$$

e.g. $\Sigma^2 \subset \mathbb{R}^3$ regular surface $\qquad \iota : \Sigma \to \mathbb{R}^3$, $p \mapsto p$. \iota

Claim: ι is an immersion.

Proof: $\Sigma \xrightarrow{\ \iota\ } \mathbb{R}^3 \xleftarrow{id} \mathbb{R}^3$.
$F(u,v) = (x(u,v), y(u,v), z(u,v))$

$id^{-1}\circ\iota\circ F(u,v) = id^{-1}\circ\iota(x(u,v), y(u,v), z(u,v))$
$= id^{-1}(x(u,v), y(u,v), z(u,v))$
$= (x(u,v), y(u,v), z(u,v))$

$$D(id^{-1}\circ\iota\circ F) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} \Rightarrow \iota_{*p} \text{ is injective at } \forall p\in\Sigma$$
$\Downarrow$
ι is an immersion on Σ.

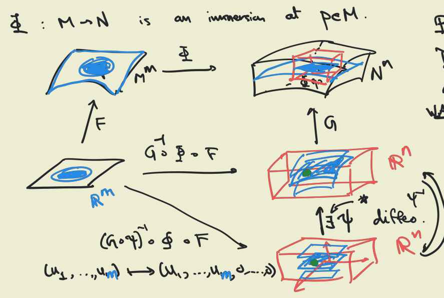
$\frac{\partial F}{\partial v}$.

$\frac{\partial F}{\partial u} = \frac{\partial x}{\partial u}\hat{i} + \frac{\partial y}{\partial u}\hat{j} + \frac{\partial z}{\partial u}\hat{k}$

(3) $\left\{\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v}\right\}$ linearly indep.

Immersion Theorem:

$\Phi : M \to N$ is an immersion at $p\in M$.



Proof:
Define
$\psi : \mathbb{R}^n \to \mathbb{R}^n$
WANT:
$\psi(u_1,\dots,u_m,0,\dots,0)$
$= G^{-1}\circ\Phi\circ F(u_1,\dots,u_m)$

Take:
$\psi(u_1,\dots,u_m,u_{m+1},\dots,u_n)$
$= G^{-1}\circ\Phi\circ F(u_1,\dots,u_m) + (0,\dots,0,u_{m+1},\dots,u_n)$

$$D\psi = \begin{bmatrix} D(G^{-1}\circ\Phi\circ F) & \begin{matrix} O \\ I \end{matrix} \end{bmatrix}$$
$\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_m}$

Φ_* is injective.

$$D(G^{-1}\circ\Phi\circ F) = \underbrace{\begin{bmatrix} \ \ \ \end{bmatrix}}_{m} \Big\} n \geq m.$$

after swapping $(u_1,\dots,u_m)$
$$\begin{bmatrix} I \\ \hline * \end{bmatrix}$$

$$\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 5 & 7 \\ 0 & 0 & 1 \end{matrix} \sim \begin{bmatrix} I \\ \hline * \end{bmatrix}.$$

e.g. $f : \mathbb{R}^3 \to \mathbb{R}$ smooth.

Given $\nabla f(x_0,y_0,z_0) \neq \vec{0}$.
$$\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \neq 0$$
at (x_0,y_0,z_0).

$\hat{i}$ $\quad \hat{\imath}$

$id_{\mathbb{R}}^{-1}\circ f\circ id_{\mathbb{R}^3}(x,y,z) = f(x,y,z)$.

$$[f_*] = D(id_{\mathbb{R}}^{-1}\circ f\circ id_{\mathbb{R}^3}) = Df = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}.$$
$\frac{\partial}{\partial x}$ $\frac{\partial}{\partial y}$ $\frac{\partial}{\partial z}$

$\neq 0$ at (x_0,y_0,z_0)

$\Rightarrow f_{*(x_0,y_0,z_0)}$ is surjective

$\Leftrightarrow f$ is a submersion at (x_0,y_0,z_0).

$$[T] \sim \begin{bmatrix} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \end{bmatrix} \to T \text{ surjective.}$$

e.g. $\Phi : \mathbb{C}^2\setminus\{0\} \to \mathbb{CP}^1$

$F(x_1,y_1,x_2,y_2) := (x_1+iy_1, x_2+iy_2)$ $\qquad$ $\Phi(z_1,z_2) := [z_1:z_2]$. $\qquad$ $G(u,v) := [1 : u+iv]$.

$G^{-1}\circ\Phi\circ F(x_1,y_1,x_2,y_2)$ $\qquad$ $x_1+y_1i \neq 0$

$= G^{-1}\circ\Phi(x_1+y_1i, x_2+y_2i)$

$= G^{-1}[x_1+y_1i : x_2+y_2i] = G^{-1}\left(\left[1 : \frac{x_2+y_2i}{x_1+y_1i}\right]\right)$

$= \left(\mathrm{Re}\left(\frac{x_2+y_2i}{x_1+y_1i}\right), \mathrm{Im}\left(\frac{x_2+y_2i}{x_1+y_1i}\right)\right)$

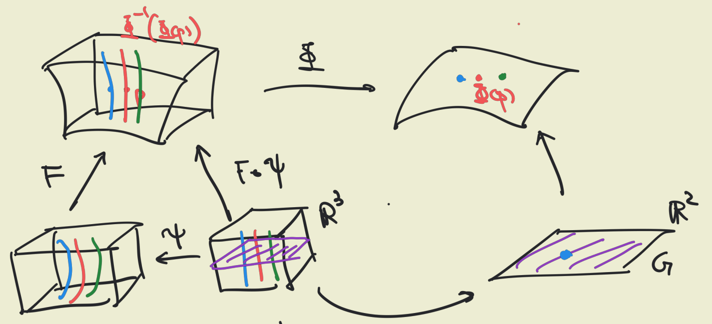
$= \left(\frac{1}{2}\left(\frac{x_2+y_2i}{x_1+y_1i} + \frac{x_2-y_2i}{x_1-y_1i}\right), \frac{1}{2i}\left(\frac{x_2+y_2i}{x_1+y_1i} - \frac{x_2-y_2i}{x_1-y_1i}\right)\right)$

$$D(G^{-1}\circ\Phi\circ F) = \begin{bmatrix} \mathrm{Re}(Z) & -\mathrm{Im}(Z) & \mathrm{Re}(W) & -\mathrm{Im}(W) \\ \mathrm{Im}(Z) & \mathrm{Re}(Z) & \mathrm{Im}(W) & \mathrm{Re}(W) \end{bmatrix}$$

$Z \neq 0$ $\qquad$ $\frac{\partial}{\partial x_1}$ $\qquad$ det $= |Z|^2 \neq 0$

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}$$

$\Phi_{*(x_1,y_1,x_2,y_2)}$ is surjective when $(x_1+y_1i) \neq 0$.
$(x_1,y_1) \neq (0,0)$.



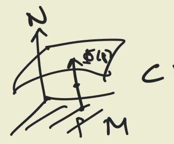
$\exists\psi$ s.t. $G^{-1}\circ\Phi\circ(F\circ\psi)(u_1,u_2,u_3)$
$= (u_1,u_2)$

Submanifolds:

$N \subset M$ is a submfd of M $\Leftrightarrow$ $\begin{cases} N \text{ is } C^\infty\text{-manifold} \\ \iota : N \to M \text{ is an immersion on } N. \\ \ \ p \mapsto p \end{cases}$
given C^∞-mfd $\qquad$ inclusion.

e.g. $\Phi : M \to N$
C^∞-mfd.

$\Gamma_\Phi := \{(x, \Phi(x)) \in M\times N : x\in M\}$.



$\subset M\times N$.

Claim: Γ_Φ is a submfd of $M\times N$.

$F \nearrow M\times N \nwarrow G$

Proof: $\iota : \Gamma_\Phi \to M\times N$.
$\tilde{F}$ $\qquad$ $F\times G$.

$\tilde{F}(u_1,\dots,u_m) := (F(u_1,\dots,u_m), \Phi(F(u_1,\dots,u_m)))$

$(F\times G)^{-1}\circ\iota\circ\tilde{F}(u_1,\dots,u_m) = (F\times G)^{-1}\circ\iota(F(u_1,\dots,u_m), \Phi(F(u_1,\dots,u_m)))$
$= (F\times G)^{-1}(F(u_1,\dots,u_m), \Phi(F(u_1,\dots,u_m)))$
$= (u_1,\dots,u_m, G^{-1}\circ\Phi\circ F(u_1,\dots,u_m))$

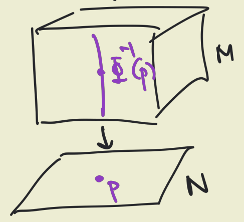
$\Rightarrow D((F\times G)^{-1}\circ\iota\circ\tilde{F})$
$$= \begin{bmatrix} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline D(G^{-1}\circ\Phi\circ F) \end{bmatrix} \to \begin{bmatrix} I \\ \hline O \end{bmatrix} \Rightarrow \iota_{*p} \text{ is injective at any } p\in M$$

e.g. $\Sigma^2 \subset \mathbb{R}^3$ is a submfd of $\mathbb{R}^3$.
regular surface
$\left(\begin{array}{l} \iota_{*p} : T_p\Sigma \to T_p\mathbb{R}^3 \text{ injective} \\ \Leftrightarrow \left\{\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v}\right\} \text{ is linearly indep.} \end{array}\right)$

Prop. 2.5.2

$\Phi : M^m \to N^n$ C^∞ map.

Given $p\in N$, consider $\Phi^{-1}(p) = \{x\in M : \Phi(x) = p\}$
assume $\Phi^{-1}(p) \neq \emptyset$.



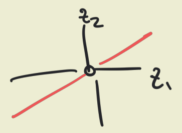
If Φ is a submersion on $\Phi^{-1}(p) \subset M$.

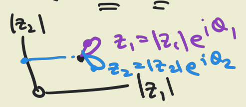
then $\Phi^{-1}(p)$ is a submanifold of M with $\dim \Phi^{-1}(p) = \dim M - \dim N$.

e.g. $\Phi : \mathbb{C}^2\setminus\{0\} \to \mathbb{CP}^1$
$(z_1,z_2) \mapsto [z_1:z_2]$.

Φ is submersion at any $(z_1,z_2) \in \mathbb{C}^2\setminus\{0\}$.

submfds of $\mathbb{C}^2\setminus\{0\}$
$$\begin{cases} \Phi^{-1}([1:0]) = \{(\lambda,0) : \lambda\in\mathbb{C}\setminus\{0\}\} \leftrightarrow \mathbb{C}\setminus\{0\}. \\ \Phi^{-1}([0:1]) = \{(0,\lambda) : \lambda\neq 0\} \leftrightarrow \mathbb{C}\setminus\{0\} \\ \Phi^{-1}([z_1:z_2]) = \{(\lambda z_1, \lambda z_2) : \lambda\neq 0\} \leftrightarrow \mathbb{C}\setminus\{0\}. \end{cases}$$

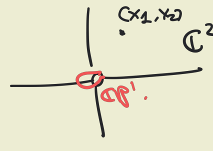




$\mathbb{C}^3\setminus\{0\} \xrightarrow{\ \psi\ } \mathbb{CP}^2$.
$(z_1,z_2,z_3) \mapsto [z_1:z_2:z_3]$.

$\psi^{-1}([1:0:0]) = \{(\lambda,0,0) : \lambda\neq 0\} \leftrightarrow \mathbb{C}\setminus\{0\}$.

e.g. $M = \{((x_1,x_2),[y_1:y_2]) \in \mathbb{C}^2\times\mathbb{CP}^1 : x_1y_2 = x_2y_1\}$.
$= \{((0,0),[y_1:y_2]) : [y_1:y_2]\in\mathbb{CP}^1\}$
$\amalg \{(\underbrace{(x_1,x_2)}_{\neq(0,0)}, [x_1:x_2]) : (x_1,x_2)\neq(0,0)\}$



$M \ni ((x_1,x_2),[y_1:y_2]) \leftarrow \dim M = 4$
$\pi_1 \swarrow \quad \searrow \pi_2$
$(x_1,x_2)\in\mathbb{C}^2 \qquad [y_1:y_2]\in\mathbb{CP}^1$.
$\dim\mathbb{C}^2 = 4$

$$\begin{array}{ccc} M & \xrightarrow{\pi_1} & \mathbb{C}^2 \\ \downarrow \pi_2 & & \\ \mathbb{CP}^1 & & \end{array}$$

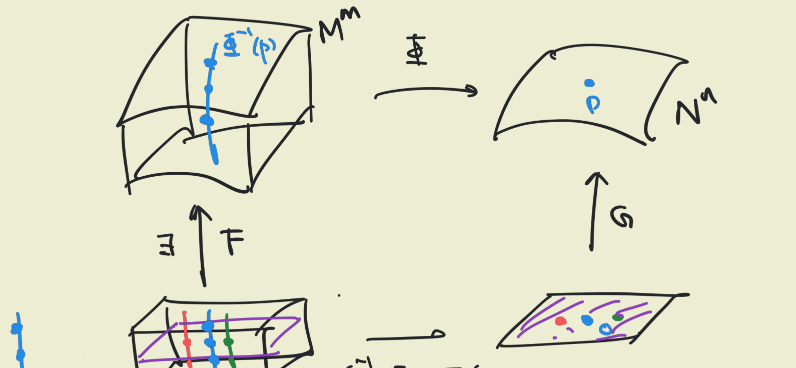
$\pi_1^{-1}((0,0)) = \{((0,0),[y_1:y_2]) : [y_1:y_2]\in\mathbb{CP}^1\}$

$\dim \pi_1^{-1}((0,0)) = \dim\mathbb{CP}^1 = 2$.

e.g. $M := \{[z_0:\dots:z_4]\in\mathbb{CP}^4 : z_0^5 + \dots + z_4^5 = 0\}$.
$\subset \mathbb{CP}^4$.
quintic Calabi-Yau 3-mfd.

$\Phi([z_0:\dots:z_4]) \cancel{=} \dfrac{z_0^5 + \dots + z_4^5}{\cancel{|z_0|^5 + \dots + |z_4|^5}}$.

$U_i \xrightarrow{F_i} \mathbb{CP}^4$ $\qquad$ $M \cap F_i(U_i)$



$G^{-1}\circ\Phi\circ F(u_1,\dots,u_n,u_{n+1},\dots,u_m)$
$= (u_1,\dots,u_n)$

$\tilde{F}(u_{n+1},\dots,u_m) := F(\underbrace{0,\dots,0}_{n}, u_{n+1},\dots,u_m)$

$\iota : \Phi^{-1}(p) \to M$ $\quad$ $F^{-1}\circ\iota\circ\tilde{F}(u_{n+1},\dots,u_m) = F^{-1}\circ\iota(F(0,\dots,0,u_{n+1},\dots,u_m))$

$= (0,\dots,0,u_{n+1},\dots,u_m)$

$$D(F^{-1}\circ\iota\circ\tilde{F}) = \begin{bmatrix} O \\ \hline I \end{bmatrix}$$