Diffeomorphism:

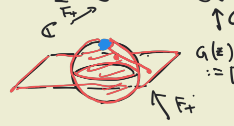
$\Phi : M \to N$ between C^∞-manifolds.

We say Φ is a diffeomorphism $\overset{\text{def}}{\iff}$ Φ is bijective and both Φ, Φ^{-1} are C^∞.

$\mathbb{CP}^1 = \{[1:w] : w \in \mathbb{C}\} \sqcup \{[0:1]\}$

$S^2 = \{x^2+y^2+z^2=1\} \subset \mathbb{R}^3$.

$\Phi : S^2 \to \mathbb{CP}^1$, $\mathbb{C} \xrightarrow{F_+} S^2$, $G : \mathbb{C} \to \mathbb{CP}^1$, $G(z) := [z:1]$



$\Phi(\vec{p}) = \begin{cases} [1 : \underbrace{F_+^{-1}(p)}_{\text{in } \mathbb{C}}] & \text{if } p \neq (0,0,1) \\ [0:1] & \text{if } p = (0,0,1) \end{cases}$ (stereographic, $\in S^1$)

$G^{-1} \circ \Phi \circ \underbrace{F_+(z)}_{\neq (0,0,1)} = G^{-1}([1 : F_+^{-1}(F_+(z))]) = G^{-1}([1:z])$

$= G^{-1}([\frac{1}{z} : 1])$ $\{z \neq 0\}$ domain.

$= \frac{1}{z}$.

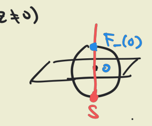
$G^{-1} \circ \Phi \circ F_-(z) = G^{-1} \circ \Phi \circ F_+ \circ (F_+^{-1} \circ F_-)(z)$

$= G^{-1} \circ \Phi \circ F_+ (\frac{1}{\bar{z}})$

$= \bar{z}$. $(z \neq 0)$

$G^{-1} \circ \Phi \circ F_-(0) = G^{-1} \circ \Phi(\text{north pole})$

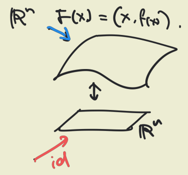
$= G^{-1}([0:1]) = 0$.



e.g. $f : \mathbb{R}^n \to \mathbb{R}$ is continuous. $\mathbb{R}^n$ $F(x) = (x, f(x))$.

$\Sigma_f := \{(x, f(x)) : x \in \mathbb{R}^n\}$

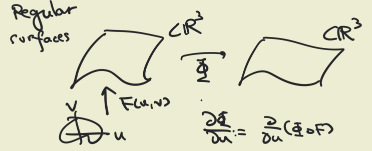
$\Phi : \mathbb{R}^n \to \Sigma_f$, $x \mapsto (x, f(x))$ (id, F)



$F^{-1} \circ \Phi \circ id(x) = F^{-1}(\Phi(x)) = F^{-1}((x, f(x))) = x$

$id^{-1} \circ \Phi^{-1} \circ F = (F^{-1} \circ \Phi \circ id)^{-1} = id$.

Regular surfaces



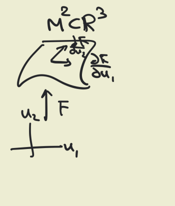
$\frac{\partial \Phi}{\partial u} := \frac{\partial}{\partial u}(\Phi \circ F)$

M^n C^∞-manifold.

Regular surfaces $\rightsquigarrow$ C^∞-manifold

$\frac{\partial F}{\partial u_i} \longrightarrow \frac{\partial}{\partial u_i}$

$\sum_i a_i \frac{\partial F}{\partial u_i} \longrightarrow \sum_i a_i \frac{\partial}{\partial u_i}$



$T_pM = \text{span}\left\{\frac{\partial}{\partial u_i}(p)\right\}_{i=1}^n$

$\frac{\partial}{\partial u_i}(p) : C^\infty(M, \mathbb{R}) \to \mathbb{R}$

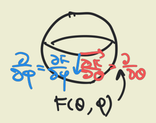
$f \mapsto \frac{\partial f}{\partial u_i}(p) = \frac{\partial}{\partial u_i}(f \circ F)$

$F(u_1, \ldots, u_n) : \mathcal{U} \to M$, $\ni p$.

$\sum_i a_i \frac{\partial}{\partial u_i}(p) : C^\infty(M, \mathbb{R}) \to \mathbb{R}$

$f \mapsto \sum_i a_i \frac{\partial f}{\partial u_i}(p)$.

$\frac{\partial}{\partial x} \leftrightarrow \frac{\partial(id)}{\partial x} = (1, 0, 0, 0)$.



$\frac{\partial}{\partial \varphi} = \frac{\partial F}{\partial \varphi}$, $\frac{\partial F}{\partial \theta} = \frac{\partial}{\partial \theta}$, $F(\theta, \varphi)$

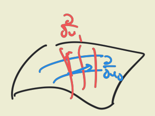
$\frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial \varphi}$

$\hat{j} = \frac{\partial}{\partial y}$, $\hat{i} = \frac{\partial}{\partial x}$, $\mathbb{R}^2$

$\mathbb{RP}^2$

$\uparrow F(u_0, u_1) := [u_0 : u_1 : 1]$

$\frac{\partial}{\partial u_0}, \frac{\partial}{\partial u_1}$



Claim: $\text{span}\left\{\frac{\partial}{\partial u_i}(p)\right\}_{i=1}^n$ is indep. of local parametrizat'ns.

$F(u_1, \ldots, u_n)$, $G(v_1, \ldots, v_n)$.

$\frac{\partial}{\partial u_i}$ $\qquad$ $\frac{\partial}{\partial v_j}$

$f \mapsto \frac{\partial}{\partial u_i}(f \circ F)$ $\qquad$ $f \mapsto \frac{\partial}{\partial v_j}(f \circ G)$

$(u_1, \ldots, u_n) \xrightarrow{G^{-1} \circ F} (v_1, \ldots, v_n)$

$\text{span}\left\{\frac{\partial}{\partial u_i}\right\}$ $\qquad$ $\text{span}\left\{\frac{\partial}{\partial v_j}\right\}$.

$(f \circ G) \circ (G^{-1} \circ F)$ — v_j — u_i

$\frac{\partial f}{\partial u_i} = \frac{\partial}{\partial u_i}(f \circ F) = \frac{\partial}{\partial u_i}(f \circ G) \circ (G^{-1} \circ F)$

$= \sum_{j=1}^n \frac{\partial (f \circ G)}{\partial v_j} \frac{\partial v_j}{\partial u_i} = \sum_{j=1}^n \frac{\partial f}{\partial v_j} \frac{\partial v_j}{\partial u_i}$

$\Rightarrow \frac{\partial}{\partial u_i} = \sum_{j=1}^n \frac{\partial v_j}{\partial u_i} \frac{\partial}{\partial v_j}$

$\Rightarrow \text{span}\left\{\frac{\partial}{\partial u_i}\right\} \subset \text{span}\left\{\frac{\partial}{\partial v_j}\right\}$

Claim: $\dim \text{span}\left\{\frac{\partial}{\partial u_i}(p)\right\} = n = \dim M$.

Proof: $\sum_{i=1}^n a_i \frac{\partial}{\partial u_i}(p) = 0$

$\forall j$: $\sum_{i=1}^n a_i \frac{\partial u_j}{\partial u_i}(p) = 0$

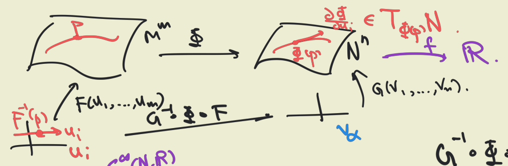
$\Rightarrow \sum_{i=1}^n a_i \delta_{ij} = 0$

$\Rightarrow a_j = 0 \quad \forall j$. ∎

$u_j : M \to \mathbb{R}$, $p \mapsto u_j$'s coord. of p.

$\text{span}\left\{\frac{\partial}{\partial u_i}(p)\right\} = \text{span}\left\{\frac{\partial}{\partial v_j}(p)\right\}$.

$du^i\left(\frac{\partial}{\partial u_j}\right) = \delta_{ij}$.



$M^m \xrightarrow{\Phi} N^n \xrightarrow{f} \mathbb{R}$, $\frac{\partial \Phi}{\partial u_i} \in T_{\Phi(p)}N$, $F(u_1, \ldots, u_m)$, $G(v_1, \ldots, v_n)$, $F^{-1}(p)$, u_i, $G^{-1} \circ \Phi \circ F$, v_α

$G^{-1} \circ \Phi \circ F(u_1, \ldots, u_n)$

$= (v_1(u_1, \ldots, u_n), \ldots)$

$\frac{\partial \Phi}{\partial u_i}\Big|_{\Phi(p)} f := \frac{\partial}{\partial u_i}(f \circ \Phi \circ F)\Big|_{F^{-1}(p)}$, $f \in C^\infty(N, \mathbb{R})$

$(f \circ G) \circ (G^{-1} \circ \Phi \circ F)$ — v_α — u_i

$= \frac{\partial}{\partial u_i}((f \circ G) \circ (G^{-1} \circ \Phi \circ F))$

$= \sum_{\alpha=1}^n \frac{\partial (f \circ G)}{\partial v_\alpha} \frac{\partial v_\alpha}{\partial u_i}$

$= \sum_{\alpha=1}^n \frac{\partial f}{\partial v_\alpha} \frac{\partial v_\alpha}{\partial u_i}$

$\Rightarrow \frac{\partial \Phi}{\partial u_i} = \sum_{\alpha=1}^n \frac{\partial v_\alpha}{\partial u_i} \frac{\partial}{\partial v_\alpha}$.

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ex. $\Phi : \mathbb{RP}^1 \times \mathbb{RP}^2 \to \mathbb{RP}^5$

$\Phi([x_0 : x_1], [y_0 : y_1 : y_2]) := [x_0y_0 : x_0y_1 : x_0y_2 : x_1y_0 : x_1y_1 : x_1y_2]$.

$\mathbb{RP}^1 \times \mathbb{RP}^2 \xrightarrow{\Phi} \mathbb{RP}^5$

$F(u) = [1:u]$, $G(v_1, v_2) = [1 : v_1 : v_2]$, $H(w_1, \ldots, w_5) = [1 : w_1 : \cdots : w_5]$.

$H^{-1} \circ \Phi \circ (F \times G)(u, v_1, v_2)$

$= H^{-1} \circ \Phi([1:u], [1 : v_1 : v_2])$

$= H^{-1}([1 : v_1 : v_2 : u : uv_1 : uv_2])$

$= (v_1, v_2, u, uv_1, uv_2) \xrightarrow{\frac{\partial}{\partial u}} (0, 0, 1, v_1, v_2)$ (w_1, w_2, w_3, w_4, w_5)

$\xrightarrow{\frac{\partial}{\partial v_1}} (1, 0, 0, u, 0)$

$\frac{\partial \Phi}{\partial u} = 1 \frac{\partial}{\partial w_3} + v_1 \frac{\partial}{\partial w_4} + v_2 \frac{\partial}{\partial w_5}$

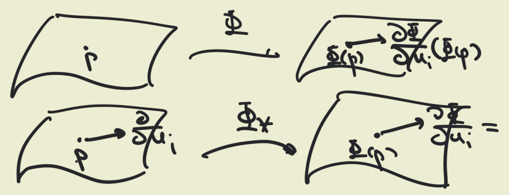
$\frac{\partial \Phi}{\partial v_1} = 1 \frac{\partial}{\partial w_1} + u \frac{\partial}{\partial w_4}$.

$\frac{\partial \Phi}{\partial v_1} \in T(\mathbb{RP}^5)$.

$\Phi_*\Big(\underbrace{\frac{\partial}{\partial v_1}}_{\in T(\mathbb{RP}^1 \times \mathbb{RP}^2)}\Big) := \frac{\partial \Phi}{\partial v_1} \in T(\mathbb{RP}^5)$

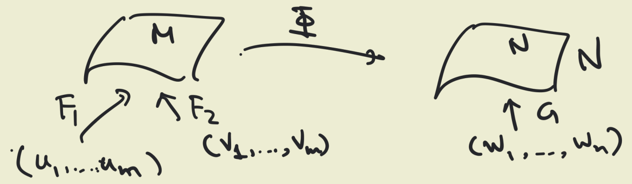
$\Phi : M \to N$, $F(u_1, \ldots, u_m)$, $G(v_1, \ldots, v_n)$

$\Phi_*\left(\frac{\partial}{\partial u_i}\right) = \frac{\partial \Phi}{\partial u_i}$



$\frac{\partial \Phi}{\partial u_i}(\Phi(p))$, $\frac{\partial \Phi}{\partial u_i} = \Phi_*\left(\frac{\partial}{\partial u_i}\right)$.

$\Phi_{*_p} : T_pM \to T_{\Phi(p)}N$ tangent map.



F_1, F_2, $(u_1, \ldots, u_m)$, $(v_1, \ldots, v_m)$, G, $(w_1, \ldots, w_n)$

WANT: Φ_* is indep. of local coordinates of M.

$X \in T_pM$, $X = \sum_i X^i \frac{\partial}{\partial u_i} = \sum_j \tilde{X}^j \frac{\partial}{\partial v_j}$

$= \sum_j \tilde{X}^j \left(\sum_i \frac{\partial}{\partial u_i} \frac{\partial u_i}{\partial v_j}\right)$

$= \sum_i \left(\sum_j \left(\frac{\partial u_i}{\partial v_j} \tilde{X}^j\right)\right) \frac{\partial}{\partial u_i}$

$\Phi_*\left(\sum_i X^i \frac{\partial}{\partial u_i}\right) = \sum_i X^i \Phi_*\left(\frac{\partial}{\partial u_i}\right)$

$= \sum_i X^i \frac{\partial \Phi}{\partial u_i}$

$\Phi_*\left(\sum_j \tilde{X}^j \frac{\partial}{\partial v_j}\right) = \sum_j \tilde{X}^j \frac{\partial \Phi}{\partial v_j}$

Need: $\sum_i X^i \frac{\partial \Phi}{\partial u_i} \cdot f = \sum_j \tilde{X}^j \frac{\partial \Phi}{\partial v_j} \cdot f \quad \forall f \in C^\infty(N)$

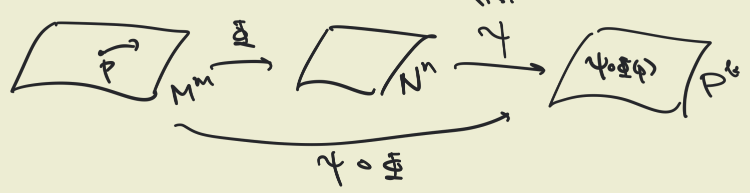
LHS $= \sum_i X^i \frac{\partial}{\partial u_i}(f \circ \Phi \circ F)$

$= \sum_i \sum_j \frac{\partial u_i}{\partial v_j} \tilde{X}^j \cdot \sum_k \frac{\partial}{\partial v_k}(f \circ \Phi \circ F) \frac{\partial v_k}{\partial u_i}$

$= \sum_{j,k} \underbrace{\frac{\partial v_k}{\partial v_j}} \tilde{X}^j \frac{\partial}{\partial v_k}(f \circ \Phi \circ F)$

$= \sum_{j,k} \delta_{jk} \tilde{X}^j \frac{\partial \Phi}{\partial v_k} \cdot f = \sum_j \tilde{X}^j \frac{\partial \Phi}{\partial v_j} \cdot f$.

$=$ RHS.



\Phi, \Psi, $M^m \xrightarrow{\Phi} N^n \xrightarrow{\Psi} P^k$, $\Psi \circ \Phi(p)$, $\Psi \circ \Phi$

$(\Psi \circ \Phi)_{*_p} : T_pM \longrightarrow T_{\Psi(\Phi(p))}P$.

chain rule: $(\Psi \circ \Phi)_{*_p} = \Psi_{*_{\Phi(p)}} \circ \Phi_{*_p}$ $\qquad$ $\boxed{(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*}$

$p \in T_pM \xrightarrow{\Phi_*} \Phi(p) \in T_{\Phi(p)}N \xrightarrow{\Psi_*} \Psi(\Phi(p)) \in T_{\Psi(\Phi(p))}P$.

Cor: If $M \xrightarrow{\Phi} N$ is diffeo.

then $T_pM \xrightarrow{\Phi_{*_p}} T_{\Phi(p)}N$ is an isomorphism. $\forall p$

Proof: $\Phi^{-1} \circ \Phi = id_M$.

$(\Phi^{-1} \circ \Phi)_* = (id_M)_* = id_{TM}$

$\Phi^{-1}_* \circ \Phi_*$

Similarly: $\Phi_* \circ \Phi^{-1}_* = id_{TN}$.

$M \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} N$

Proof of chain rule:

$M^m \xrightarrow{\Phi} N^n \xrightarrow{\Psi} P^k$, F, $(u_1, \ldots, u_m)$, G, $(v_1, \ldots, v_n)$, H, $(w_1, \ldots, w_k)$

$(\Psi \circ \Phi)_*\left(\frac{\partial}{\partial u_i}\right)$

$= \frac{\partial}{\partial u_i}(\Psi \circ \Phi)$

$= \sum_j \frac{\partial w_j}{\partial u_i} \frac{\partial}{\partial w_j}$

$\Psi_* \circ \Phi_*\left(\frac{\partial}{\partial u_i}\right)$

$= \Psi_*\left(\frac{\partial \Phi}{\partial u_i}\right) = \Psi_*\left(\sum_k \frac{\partial v_k}{\partial u_i} \frac{\partial}{\partial v_k}\right)$

$= \sum_k \frac{\partial v_k}{\partial u_i} \Psi_*\left(\frac{\partial}{\partial v_k}\right)$

$= \sum_k \frac{\partial v_k}{\partial u_i} \sum_j \frac{\partial w_j}{\partial v_k} \frac{\partial}{\partial w_j}$

$= \sum_j \frac{\partial w_j}{\partial u_i} \frac{\partial}{\partial w_j}$.

$\Phi(t) : \mathbb{R} \longrightarrow S^1 \subset \mathbb{R}^2$.

$t \longmapsto (\cos t, \sin t)$

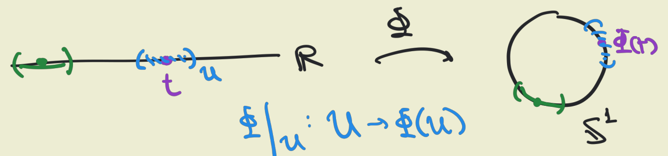
$\Phi_{*_t} : \underset{1d}{T_t\mathbb{R}} \to \underset{1d}{T_{\Phi(t)}S^1}$.

$\frac{\partial \Phi}{\partial t} = (-\sin t, \cos t) \neq 0$.

$\Phi_*\left(\frac{\partial}{\partial t}\right)$ $\qquad$ $[\neq 0]$

$\Rightarrow \Phi_{*_t} = [\neq 0]$ is invertible.

But $\mathbb{R}$ and S^1 not diffeomorphism.



$\mathbb{R}$, t, $\mathcal{U}$, Φ, $\Phi(t)$, S^1

$\Phi|_\mathcal{U} : \mathcal{U} \to \Phi(\mathcal{U})$ is diffeomorphism

$\Phi : M \to N$ is a local diffeomorphism

$\overset{\text{def}}{\iff}$ $\forall p \in M$, $\exists$ open $\mathcal{U}_{\ni p} \subset M$ and

$\exists$ open $V_{\ni \Phi(p)} \subset N$

s.t. $\Phi|_\mathcal{U} : \mathcal{U} \to V$ is a diffeomorphism.

Helicoid $\qquad$ $\mathbb{R}^2 \setminus \{(0,0)\}$.

locally diffeo, π

Helix: $(\cos t, \sin t, t)$