Examples of manifolds:

① Regular surfaces

② Regular hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$

$F \nearrow \ \nwarrow G$, $U \subset \mathbb{R}^n$

$\left\{ \frac{\partial F}{\partial u_1}, \dots, \frac{\partial F}{\partial u_n} \right\}$ linearly indep.

③ M can be parametrized by one single chart $F: U \subset \mathbb{R}^n \xrightarrow{1-1} M$ homeomorphic.

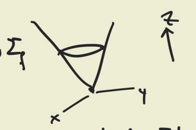
$F^{-1} \circ F(x) = x$

$\Sigma_f := \{ (x, f(x)) : x \in \mathbb{R}^n \}$ $f: \mathbb{R}^n \to \mathbb{R}$ continuous.

is a C^∞-manifold

$F(x) := (x, f(x))$.

$f(x,y) = \sqrt{x^2+y^2}$. $\longrightarrow \Sigma_f$



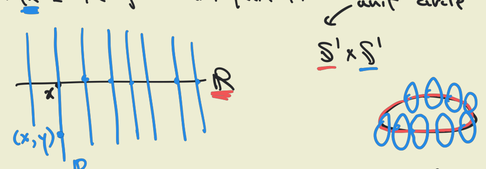
Yes! It is a C^∞ manifold.

④ M_1^k, M_2^l given: C^∞-manifolds

$M_1 \times M_2 := \{ (x,y) : x \in M_1, y \in M_2 \}$

e.g. $\mathbb{R} \times \mathbb{R} = \{ (x,y) : x \in \mathbb{R}, y \in \mathbb{R} \}$.

unit circle $S^1 \times S^1$



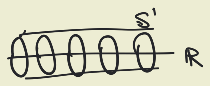
torus.

$F_\alpha : U_\alpha \subset \mathbb{R}^k \to M_1$, $G_\beta : V_\beta \subset \mathbb{R}^l \to M_2$, $\widetilde{F}_\alpha \times \widetilde{G}_\beta$, $F_\alpha \times G_\beta : U_\alpha \times V_\beta \to M_1 \times M_2$

$(F_\alpha \times G_\beta)(u,v) := (F_\alpha(u), G_\beta(v)) \in M_1 \times M_2$, $u \in U_\alpha$, $v \in V_\beta$, $F_\alpha(u) \in M_1$, $G_\beta(v) \in M_2$

$(\widetilde{F}_\alpha \times \widetilde{G}_\beta)^{-1} \circ (F_\alpha \times G_\beta)(u,v)$
$= (\widetilde{F}_\alpha \times \widetilde{G}_\beta)^{-1} (F_\alpha(u), G_\beta(v))$
$= \left(\underbrace{\widetilde{F}_\alpha^{-1} \circ F_\alpha(u)}_{C^\infty}, \underbrace{\widetilde{G}_\beta^{-1} \circ G_\beta(v)}_{C^\infty} \right)$ is C^∞.

$\mathbb{R} \times S^1$:



Quotient set.

$\mathbb{R}$ $x \sim y \overset{\text{def}}{\iff} x - y \in \mathbb{Z}$.

$[x] := \{ y \in \mathbb{R} : y \sim x \}$

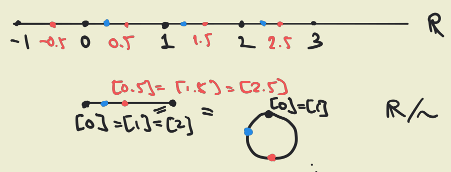
$[1] = \{ 0, 1, 2, 3, 4, \dots, -1, -2, -3, -4, \dots \}$

$[2] = \{ 0, 1, 2, 3, 4, \dots, -1, -2, -3, -4, \dots \}$

$[1] = [2] = [3] = \dots$

$[0.5] = \{ 0.5, 1.5, 2.5, 3.5, \dots, -0.5, -1.5, \dots \} \neq [1]$.

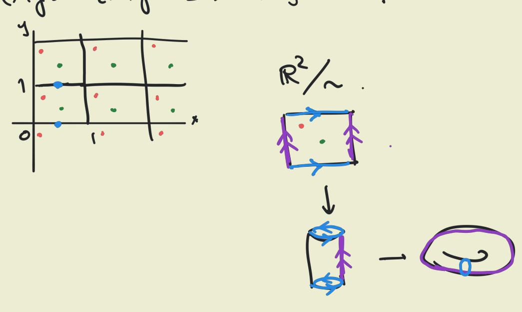
$\mathbb{R}/\sim := \{ [x] : x \in \mathbb{R} \}$



$[0.5] = [1.5] = [2.5]$, $[0] = [1] = [2]$, $\mathbb{R}/\sim$

$\mathbb{R}^2 = \{ (x,y) : x, y \in \mathbb{R} \}$.

$(x,y) \sim (\tilde{x}, \tilde{y}) \overset{\text{def}}{\iff} (x,y) - (\tilde{x}, \tilde{y}) \in \mathbb{Z} \times \mathbb{Z}$.

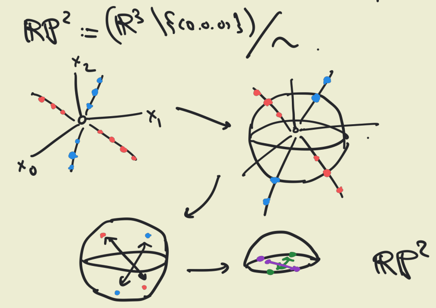


$\mathbb{R}^2/\sim$

$\mathbb{RP}^2$ real projective plane

$\mathbb{R}^3 \setminus \{(0,0,0)\}$ $(x_0, x_1, x_2) \sim (y_0, y_1, y_2) \overset{\text{def}}{\iff} \exists \lambda \neq 0$ s.t. $(x_0, x_1, x_2) = (\lambda y_0, \lambda y_1, \lambda y_2)$

$\mathbb{RP}^2 := (\mathbb{R}^3 \setminus \{(0,0,0)\}) / \sim$



$\mathbb{RP}^2$

$\mathbb{RP}^n := (\mathbb{R}^{n+1} \setminus \{\vec{0}\}) / \sim$

$\underbrace{(x_0, \dots, x_n)}_{\vec{x}} \sim \underbrace{(y_0, \dots, y_n)}_{\vec{y}} \overset{\text{def}}{\iff} \exists \lambda \neq 0$ s.t. $\vec{x} = \lambda \vec{y}$.

$[x_0, \dots, x_n] = [y_0, \dots, y_n]$

$[x_0 : \dots : x_n] := [(x_0, \dots, x_n)]$.

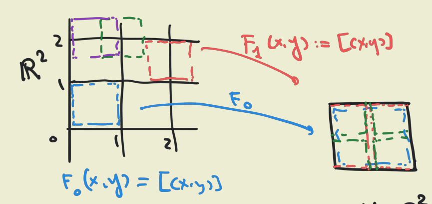
$[1 : -1 : \pi] = [-2, 2, -2\pi]$

$\mathbb{CP}^n := (\mathbb{C}^{n+1} \setminus \{\vec{0}\}) / \sim$ complex projective space.

$(z_0, \dots, z_n) \sim (w_0, \dots, w_n) \overset{\text{def}}{\iff} \exists \lambda \neq 0$ s.t. $z_i = \lambda w_i \ \forall i$. ($\lambda \in \mathbb{C}$)

$M = \{ [z_0 : z_1 : z_2 : z_3 : z_4] \in \mathbb{CP}^4 : z_0^5 + \dots + z_4^5 = 0 \}$ Calabi-Yau.

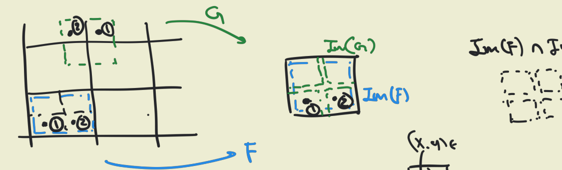
$\mathbb{R}^2/\sim$ $(x,y) \sim (\tilde{x}, \tilde{y}) \iff (x,y) - (\tilde{x}, \tilde{y}) \in \mathbb{Z} \times \mathbb{Z}$.



$F_1(x,y) := [(x,y)]$

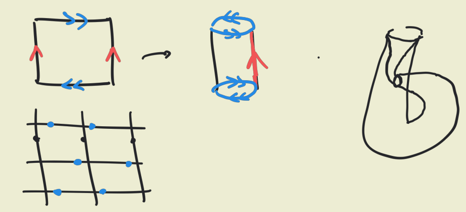
$F_0(x,y) = [(x,y)]$

$F_0 : (0,1) \times (0,1) \to M$. $M = \mathbb{R}^2/\sim$



$\mathrm{Im}(F) \cap \mathrm{Im}(G)$

$G^{-1} \circ F(x,y) = \begin{cases} (x,y) + (1,2) & \text{if } ① \\ (x,y) + (0,2) & \text{if } ② \\ (x,y) + (\ , \ ) & \text{if} \\ (x,y) + (\ , \ ) & \text{if} \end{cases}$ is C^∞ on



$\mathbb{RP}^2$ is a C^∞-manifold.

$(\mathbb{R}^3 \setminus \{\vec{0}\}) / \sim = \{ [x_0 : x_1 : x_2] : (x_0, x_1, x_2) \neq (0,0,0) \}$

$[1:2:3] = [-4:-8:-12]$

$[6:8:9] \neq [7:7:7]$.

$F_0(u_1, u_2) := [1 : u_1 : u_2]$

$F_1(v_0, v_2) := [v_0 : 1 : v_2]$

$F_2(w_0, w_1) := [w_0 : w_1 : 1]$.

$[\overset{\neq 0}{x_0} : x_1 : x_2] = [1 : \frac{x_1}{x_0} : \frac{x_2}{x_0}]$

$F_2^{-1} \circ F_0(u_1, u_2) = F_2^{-1}([1 : u_1 : u_2])$
$= F_2^{-1}([\frac{1}{u_2} : \frac{u_1}{u_2} : 1])$ $\{u_2 \neq 0\}$
$= (\frac{1}{u_2}, \frac{u_1}{u_2})$ is C^∞ on

Similarly, $F_i^{-1} \circ F_j$ are C^∞ on its domain.

$\Rightarrow \mathbb{RP}^2$ is a C^∞-manifold.

$\mathbb{RP}^1 =$ ($\mathbb{R}^2 \setminus \{(0,0)\}$)

$[x:y]$, $(x,y) \neq (0,0)$

$\theta := \frac{1}{2} \mathrm{Arg}(x+iy)$ $= S^1$.

$(\cos\theta, \sin\theta) \in S^1 \longleftrightarrow [x:y] \in \mathbb{RP}^1$.

$\mathbb{RP}^2$, $\mathbb{RP}^3 \cong SO(3)$. (FYI).

$= \{ A \in M_{3 \times 3} : A^T A = I, \det(A) = 1 \}$, $\subset \mathbb{R}^9$

$\mathbb{CP}^1 \cong S^2$

$\{ [z_0 : z_1] : (z_0, z_1) \neq (0,0) \}$, $\in \mathbb{C}^2$

$= \{ [z_0 : z_1] : z_1 \neq 0, z_0 \in \mathbb{C} \} \leftarrow \{ [\frac{z_0}{z_1} : 1] : z_0 \in \mathbb{C}, z_1 \neq 0 \}$

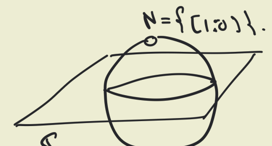
$= \{ [w : 1] : w \in \mathbb{C} \}$. $w \in \mathbb{C}$.

$\sqcup \{ [z_0 : z_1] : z_1 = 0, z_0 \neq 0 \}$

$\{ [z_0 : 0] : z_0 \neq 0 \}$

$= \{ [1:0] \}$.

$N = \{[1:0]\}$.



$\mathbb{C} \cong S^2 \setminus \{N\}$.

$M := \{ ((x_1, x_2), [y_1 : y_2]) \in \mathbb{R}^2 \times \mathbb{RP}^1 : x_1 y_2 = x_2 y_1 \}$.

$= \{ ((x_1, x_2), [y_1 : y_2]) : (x_1, x_2) \neq (0,0), x_1 y_2 = x_2 y_1 \}$ Σ_1

$\sqcup \{ ((0,0), [y_1 : y_2]) : 0 y_2 = 0 y_1 \}$ $\Sigma_0 = \{(0,0), [y_1 : y_2]\} \leftrightarrow \mathbb{RP}^1$

$((x_1, x_2), (y_1, y_2))$

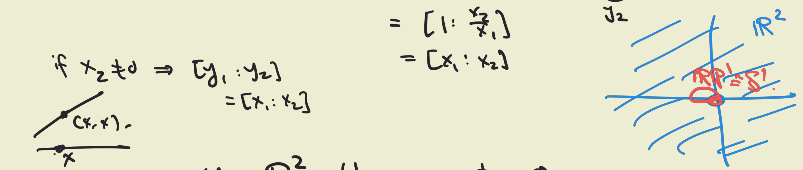
$\in \Sigma_1 = \{ ((x_1, x_2), [x_1 : x_2]) : (x_1, x_2) \neq (0,0) \} \longleftrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$.

if $x_1 \neq 0$, $\Rightarrow y_2 = \frac{x_2}{x_1} y_1 \Rightarrow [y_1 : y_2] = [y_1 : \frac{x_2}{x_1} y_1]$

$= [1 : \frac{x_2}{x_1}]$

$= [x_1 : x_2]$

if $x_2 \neq 0 \Rightarrow [y_1 : y_2] = [x_1 : x_2]$



$M = \mathbb{R}^2$ blown-up at 0.

$M = \{ ((x_1, x_2), [y_1 : y_2]) : (x_1, x_2) \in \mathbb{R}^2, x_1 y_2 = x_2 y_1 \}$.

$F_0(u_1, u_2) = ((u_1, u_1 u_2), [1 : u_2])$

$F_1(v_1, v_2) = ((v_1 v_2, v_2), [v_1 : 1])$

$F_1^{-1} \circ F_0(u_1, u_2) = (\overset{?}{v_1}, \overset{?}{v_2})$

$\iff F_0(u_1, u_2) = F_1(v_1, v_2)$

$\iff \begin{cases} \underbrace{(u_1, u_1 u_2)}_{} = \underbrace{(v_1 v_2, v_2)}_{} \\ [1 : u_2] = [v_1 : 1] = [1 : \frac{1}{v_1}] \end{cases}$ $\{u_2 \neq 0\}$.

$[\frac{1}{u_2} : 1]$, $\neq 0$, $\neq 0$

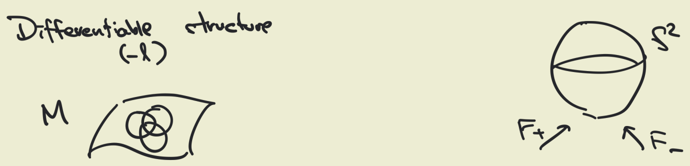
$\iff \begin{cases} v_1 = \frac{1}{u_2} \\ v_2 = \frac{u_1}{u_2} \end{cases} \Rightarrow F_1^{-1} \circ F_0(u_1, u_2) = (\frac{1}{u_2}, \frac{u_1}{u_2})$

is C^∞ on its domain $= \{ (u_1, u_2) \in \mathbb{R}^2 : u_2 \neq 0 \}$.

Similarly, $F_0^{-1} \circ F_1$ is C^∞.

$\Rightarrow M$ is a C^∞-manifold ($\dim M = 2$).

Differentiable structure (-1)



$\mathcal{A} = \{ F_\alpha : U_\alpha \to O_\alpha \subset M \}$

$\mathcal{D}(\mathcal{A}) = \{$ gigantic family of all compatible local parametrizations containing $\mathcal{A}$ $\}$.

differentiable structure generated by $\mathcal{A}$.

$\mathbb{R}^2$, $\hat{G}$, $\hat{G}(u,v) = (e^u, e^v)$, $G(u,v) = (u,v)$

$\mathbb{R}^2$, $F(u,v) = (u, v + |u|)$.

$G^{-1} \circ F$ is not C^∞

$F^{-1} \circ G$ is not C^∞.

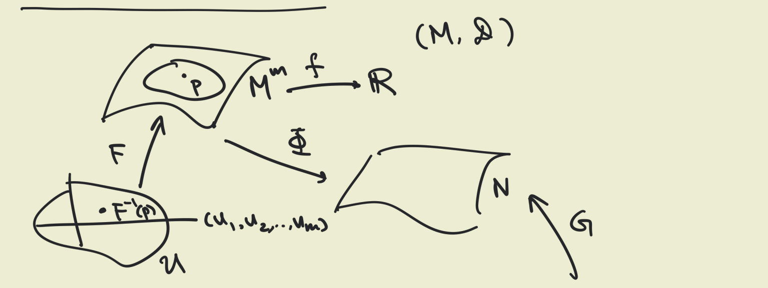
$\mathcal{D}\{F\} = \{ F, \dots \}$ $\not\ni G, \hat{G}$

$\mathcal{D}\{G\} = \{ G, \hat{G}, \dots \}$

$(\mathbb{R}^2, \mathcal{D}_1) \neq (\mathbb{R}^2, \mathcal{D}_2)$

$\mathbb{R}^2$

$(M, \mathcal{D})$



$M^m \xrightarrow{f} \mathbb{R}$, Φ, N, F, G, $F^{-1}(p)$, $(u_1, u_2, \dots, u_m)$, $\mathcal{U}$, $(v_1, \dots, v_n)$

$\frac{\partial f}{\partial u_i}(p) := \frac{\partial}{\partial u_i}(f \circ F) \Big|_{F^{-1}(p)}$.

Φ is C^k at $p \iff G^{-1} \circ \Phi \circ F$ is C^k at $F^{-1}(p)$

$\frac{\partial \Phi}{\partial u_i} \overset{?}{=} \frac{\partial}{\partial u_i}(\Phi \circ F)$ $\exists$ T.G.

$\widetilde{G}^{-1} \circ \Phi \circ \widetilde{F} = \widetilde{G}^{-1} \circ G \circ (G^{-1} \circ \Phi \circ F) \circ (F^{-1} \circ \widetilde{F})$