MATH 4033 • Spring 2019 • Calculus on Manifolds
Problem Set #2 • Abstract Manifolds • Due Date: 10/03/2019, 11:59PM

1. (30 points) Let Σ be a regular surface in $\mathbb{R}^3$ such that $(0,0,0) \notin \Sigma$. For each $p \in \Sigma$, define $N_p\Sigma$ to be the 1-dimensional vector space spanned by a non-zero normal vector to Σ at p. Consider the set:
$$N\Sigma := \{(p, n_p) \in \{p\} \times N_p\Sigma : p \in \Sigma\},$$
and the following subset of $\Sigma \times \mathbb{R}^3$:
$$L\Sigma := \{(p, tp) \in \Sigma \times \mathbb{R}^3 : t \in \mathbb{R}\}.$$
   (a) Show that $N\Sigma$ is a C^∞ 3-manifold.
   (b) Show that $L\Sigma$ is a C^∞ 3-manifold, and is a submanifold of $\Sigma \times \mathbb{R}^3$.
   (c) Suppose further that there exists a well-defined C^∞ map $\hat{\nu} : \Sigma \to \mathbb{R}^3$, where $\hat{\nu}(p)$ is a unit normal vector to Σ at p. Show that $N\Sigma$ and $L\Sigma$ are diffeomorphic.
   [Hint: If any non-zero vectors v, w in $\mathbb{R}^3$ are parallel to each other, then $v = \frac{\langle v,w\rangle}{\|w\|^2} w$.]

2. (25 points) Consider $\Phi : \mathbb{RP}^2 \to \mathbb{R}^4$ be given by
$$\Phi([x : y : z]) = \frac{(x^2 - y^2, xy, zx, yz)}{x^2 + y^2 + z^2}.$$
   (a) Show that Φ is well-defined, and is injective.
   (b) Cover $\mathbb{RP}^2$ by the standard coordinate charts. Compute the local coordinate expressions of Φ respect each coordinate charts.
   (c) Compute the matrix representation $[\Phi_*]$ with respect to **each** local coordinates chart of $\mathbb{RP}^2$.
   (d) Show that Φ is an immersion.

3. (20 points) The famous quintic Calabi-Yau 3-fold in string theory is the following subset in $\mathbb{CP}^4$:
$$M := \left\{[z_0 : z_1 : z_2 : z_3 : z_4] \in \mathbb{CP}^4 : z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0\right\}.$$
Show that M is a 6-dimensional submanifold of $\mathbb{CP}^4$ (which is 8 dimensional).
[Hint: First be careful that $\Phi([z_0 : z_1 : z_2 : z_3 : z_4]) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 : \mathbb{CP}^4 \to \mathbb{C}$ is NOT well-defined! Try to show $M \cap F_i(\mathcal{U})$ is a submanifold of $\mathbb{CP}^4$ for each i where F_i's are the standard local coordinate charts of $\mathbb{CP}^4$.]

4. (25 points) Let $p(x_1, \cdots, x_k)$ be a m-homogeneous polynomial of k variables where $k, m \geq 2$, i.e.
$$p(\lambda x_1, \cdots, \lambda x_k) = \lambda^m p(x_1, \cdots, x_k)$$
for any $\lambda > 0$, $(x_1, \cdots, x_k) \in \mathbb{R}^k$.
   (a) Prove that for any $(x_1, \cdots, x_k) \in \mathbb{R}^k$, the following identity holds:
$$\sum_{i=1}^{k} x_i \frac{\partial p}{\partial x_i}(x_1, \cdots, x_k) = mp(x_1, \cdots, x_k).$$
   (b) Show that for any $a \neq 0$, the level-set $p^{-1}(a)$, whenever non-empty, is a $(k-1)$-submanifold of $\mathbb{R}^k$.
   (c) Show that for any $a \neq 0$, we have $p^{-1}(a)$ is diffeomorphic to $p^{-1}(a/|a|)$ (or both are empty).